Introduction to Conformal Invariance for 2D Critical Lattice Models

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Contents

1 Bernoulli Bond Percolation

Suppose $G = (V, E)$ is a graph. Bernoulli bond percolation is a probability measure $\mathbb{P}_p$ on $\omega = (\omega(e) : e \in E) \in \{0,1\}^E$ for which each edge of E is open with probability p and closed with probability $1-p$, independently for different edges. The parameter $p \in [0,1]$ is called edge-weight. The σ-algebra of measurable events is the smallest σ-algebra containing events depending on finitely many edges.

1.1 Basic Properties

Consider the partial order on $\{0,1\}^E$ given by

$$\omega \le \omega' \quad \text{if and only if} \quad \omega(e) \le \omega'(e), \text{ for all } e \in \mathbb{E}.$$

This order induces a notion of increasing function from $\{0,1\}^E$ to $\mathbb{R}$. An event A is said to be increasing if its indicator function $\mathbb{1}_A$ is increasing. Note that A is increasing if for any $\omega \in A$ and $\omega' \ge \omega$, we have $\omega' \in A$.

Lemma 1.1 (Monotonicity). *Let $p \le p'$. Then for any increasing event A, we have $\mathbb{P}_p[A] \le \mathbb{P}_{p'}[A]$.*

Proof. We can couple percolation measures with different edge-weights on the same probability space as follows. Consider $\Omega = [0,1]^E$ and let $(U_e, e \in E)$ be a family of independent uniform random variables on $[0,1]$. Set P for the associated measure. For $p \in [0,1]$, introduce

$$\omega_p(e) := \mathbb{1}_{\{U_e \le p\}}\,.$$

Then it is clear that ω_p has the law of $\mathbb{P}_p$. By the construction, we have $\omega_p \le \omega_{p'}$ for any $p \le p'$. Then the claim follows trivially:

$$\mathbb{P}_p[A] = P[\omega_p \in A] \le P[\omega_{p'} \in A] = \mathbb{P}_{p'}[A].$$

□

Lemma 1.2 (FKG Inequality). *Let $p \in [0,1]$, for any two increasing functions f and g which are bounded, we have*

$$\mathbb{E}_p[fg] \ge \mathbb{E}_p[f]\mathbb{E}_p[g].$$

In particular, for any two increasing events A and B, we have

$$\mathbb{P}_p[A \cap B] \ge \mathbb{P}_p[A]\mathbb{P}_p[B].$$

Proof. Note that it is sufficient to prove the conclusion for increasing functions depending only on finitely many edges, since the general increasing function can be approximated by such functions (see Exercise 1.1). Suppose f and g are increasing functions depending on the status of the edges $\{e_1, e_2, \dots, e_N\}$. We will prove the conclusion by induction on N.

First, the conclusion is true for $N = 1$, since

$$\mathbb{E}_p[fg] = pf(1)g(1) + (1-p)f(0)g(0) \ge (pf(1) + (1-p)f(0))(pg(1) + (1-p)g(0)).$$

Next, assume the conclusion is true for $N \le n$. When $N = n+1$, we have

$$\begin{aligned}
\mathbb{E}_p[fg] &= \mathbb{E}_p\left[f(\omega(e_1), \dots, \omega(e_{n+1}))g(\omega(e_1), \dots, \omega(e_{n+1}))\right] \\
&= \mathbb{E}_p\left[\mathbb{E}_p[f(\omega(e_1), \dots, \omega(e_{n+1}))g(\omega(e_1), \dots, \omega(e_{n+1})) \,|\, \omega(e_{n+1})]\right] \\
&\ge \mathbb{E}_p\left[\mathbb{E}_p[f(\omega(e_1), \dots, \omega(e_{n+1})) \,|\, \omega(e_{n+1})]\mathbb{E}_p[g(\omega(e_1), \dots, \omega(e_{n+1})) \,|\, \omega(e_{n+1})]\right] \\
&\hspace{20em} \text{(by induction hypothesis)} \\
&\ge \mathbb{E}_p\left[\mathbb{E}_p[f(\omega(e_1), \dots, \omega(e_{n+1})) \,|\, \omega(e_{n+1})]\right] \mathbb{E}_p\left[\mathbb{E}_p[g(\omega(e_1), \dots, \omega(e_{n+1})) \,|\, \omega(e_{n+1})]\right] \\
&\hspace{20em} \text{(by the proof for } N = 1) \\
&= \mathbb{E}_p\left[f(\omega(e_1), \dots, \omega(e_{n+1}))\right] \mathbb{E}_p\left[g(\omega(e_1), \dots, \omega(e_{n+1}))\right].
\end{aligned}$$

Thus the conclusion holds for $N = n+1$. This completes the proof. □

Next, we will introduce Russo's formula. For a configuration ω, let ω^e be the configuration coinciding with ω for edges different from e, and with the edge e open; similarly, let ω_e be the configuration coinciding with ω for edges different from e, and with the edge e closed. Let A be an increasing event depending on the states of a finite set of edges E. We say that e is pivotal for A if $\omega^e \in A$ but $\omega_e \notin A$. Note that the event $\{e \text{ is pivotal for } A\}$ does not depend on the status of e.

Lemma 1.3 (Russo's formula)**.** *Let A be an increasing event depending on the states of a finite set of edges E. Then*

$$\frac{d}{dp}\mathbb{P}_p[A] = \sum_{e\in E}\mathbb{P}_p[e \textit{ is pivotal for } A].$$

Proof. We introduce the following probability measure on percolation configuration: suppose $E = \{e_1, \ldots, e_N\}$. For $\overrightarrow{p} = (p_1, \ldots, p_N) \in [0,1]^N$, let $\mathbb{P}_{\overrightarrow{p}}$ be the probability measure such that the edge e_j is open with probability p_j and closed with probability $1 - p_j$, independently for $1 \le j \le N$. Then the weight for a configuration $\omega \in \{0,1\}^E$ is given by

$$\mathbb{P}_{\overrightarrow{p}}[\omega] = \prod_j \left(p_j \, \mathbb{1}_{\{\omega(e_j)=1\}} + (1-p_j)\, \mathbb{1}_{\{\omega(e_j)=0\}}\right).$$

Thus,

$$\mathbb{P}_{\overrightarrow{p}}[A] = \sum_\omega \mathbb{1}_A(\omega)\mathbb{P}_{\overrightarrow{p}}[\omega].$$

For $1 \le n \le N$, differentiate with respect to p_n, we have

$$\begin{aligned}
\frac{\partial}{\partial p_n}\mathbb{P}_{\overrightarrow{p}}[A] &= \sum_\omega \left(\frac{1}{p_n}\mathbb{1}_{\{\omega(e_n)=1\}} - \frac{1}{1-p_n}\mathbb{1}_{\{\omega(e_n)=0\}}\right)\mathbb{1}_A(\omega)\mathbb{P}_{\overrightarrow{p}}[\omega] \\
&= \sum_\omega \left(\frac{1}{p_n}\mathbb{1}_{\{\omega(e_n)=1\}}\mathbb{1}_A(\omega^{e_n}) - \frac{1}{1-p_n}\mathbb{1}_{\{\omega(e_n)=0\}}\mathbb{1}_A(\omega_{e_n})\right)\mathbb{P}_{\overrightarrow{p}}[\omega] \\
&= \mathbb{P}_{\overrightarrow{p}}[\omega^{e_n} \in A] - \mathbb{P}_{\overrightarrow{p}}[\omega_{e_n} \in A] = \mathbb{P}_{\overrightarrow{p}}[e_n \text{ is pivotal for } A].
\end{aligned}$$

Therefore

$$\frac{d}{dp}\mathbb{P}_p[A] = \sum_n \left(\frac{\partial}{\partial p_n}\mathbb{P}_{\overrightarrow{p}}[A]\right)_{p_n=p,\forall n} = \sum_n \mathbb{P}_p[e_n \text{ is pivotal for } A].$$

□

1.2 Phase Transition

In the rest of this section, we focus on the bond percolation on the square lattice. The vertex-set $V(\mathbb{Z}^2)$ will be identified with $\mathbb{Z}^2$ and the edge-set $E(\mathbb{Z}^2)$ is composed of pairs of nearest neighbors:

$$V(\mathbb{Z}^2) = \{x = (x_1, x_2) : x_1, x_2 \in \mathbb{Z}\}, \quad E(\mathbb{Z}^2) = \{\{x, y\} : |x_1 - y_1| + |x_2 - y_2| = 1\}.$$

We denote the box of size n by $\Lambda_n := [-n, n]^2$. Consider a finite subgraph G of $\mathbb{Z}^2$, the boundary of G is given by

$$\partial G := \{x \in V(G) : \text{there exists } y \notin V(G) \text{ such that } \{x, y\} \in E(\mathbb{Z}^2)\}.$$

For the bond percolation on $\mathbb{Z}^2$, we are interested in the probability

$$\theta(p) := \mathbb{P}_p[0 \longleftrightarrow \infty].$$

If $\theta(p) > 0$, there is positive chance for the model to have infinite cluster, or *to percolate*. By Monotonicity Lemma 1.1, it is clear that $\theta(p)$ is increasing in p. We will show the following phase transition of the bond percolation.

Theorem 1.4. *For Bernoulli bond percolation on $\mathbb{Z}^2$, there exists $p_c \in (0,1)$ such that*

$$\theta(p) = 0, \quad \text{if } p < p_c; \qquad \theta(p) > 0 \quad \text{if } p > p_c.$$

In other words, there is a phase transition between a regime without infinite cluster and a regime with an infinite cluster. The $p < p_c$ regime is called subcritical; the $p > p_c$ regime is called supercritical. When $p = p_c$, one speaks of the critical regime.

Define $p_c = \sup\{p : \theta(p) = 0\}$, then we have $\theta(p) = 0$ for $p < p_c$ and $\theta(p) > 0$ for $p > p_c$. It remains to show $p_c \in (0,1)$. We will prove $p_c > 0$ and $p_c < 1$ separately.

Proof of Theorem 1.4, $p_c > 0$. To show $p_c > 0$, it is sufficient to show that there exists $p > 0$ such that $\theta(p) = 0$. Note that $\theta(p) = \mathbb{P}_p[0 \longleftrightarrow \infty]$. The event $\{0 \longleftrightarrow \infty\}$ implies that there exists an open path of length n starting from the origin for all $n \geq 1$. Let $\mathcal{L}_n$ be the collection of all paths of length n starting from the origin. We have a trivial bound: $\#\mathcal{L}_n \leq 4^n$. Thus, for any $n \geq 1$, we have

$$\theta(p) \leq \mathbb{P}_p[\exists L \in \mathcal{L}_n \text{ such that all edges along } L \text{ are open}] \leq p^n \#\mathcal{L}_n \leq (4p)^n.$$

This is true for all $n \geq 1$. Thus $\theta(p) = 0$ when $p < 1/4$. This gives a lower bound on the critical value: $p_c \geq 1/4$. $\square$

To show $p_c < 1$, we need to introduce the duality of a bond percolation configuration. Suppose $G = (V, E)$ is a plane graph[1], the dual graph of G is a graph that has a vertex for each face of G. The dual graph has an edge whenever two faces of G are separated from each other by an edge. Thus, each edge $e \in E$ of G has a corresponding dual edge, denoted by e^*, whose endpoints are the dual vertices corresponding to the faces on either side of e. The dual graph is denoted by $G^* = (V^*, E^*)$. Note that the dual graph of $\mathbb{Z}^2$ is $\mathbb{Z}^2 + (1/2, 1/2)$.

Suppose $G = (V, E)$ is a plane graph, for any configuration $\omega \in \{0,1\}^E$, define the configuration $\omega^* \in \{0,1\}^{E^*}$ by

$$\omega^*(e^*) = 1 - \omega(e), \quad \forall e \in E.$$

In words, a dual edge is open if the corresponding edge is closed, and vice versa. If the configuration ω is sampled according Bernoulli bond percolation on G with edge-weight p, then the configuration ω^* is distributed according to a Bernoulli bond percolation on G^* with edge-weight $1 - p$; see Figure 1.1.

Proof of Theorem 1.4, $p_c < 1$. In order for the origin not to be connected to infinity, the dual configuration ω^* must contain an open circuit of dual-edges surrounding the origin. For $m \geq 2n, n \geq 1$, let $\mathcal{L}_{m,n}$ be the set of dual circuits of length m surrounding the origin and passing by the point $(n + 1/2, 0)$. Then we see

$$\begin{aligned}
\mathbb{P}_p[0 \not\longleftrightarrow \infty] &= \mathbb{P}_p\left[\exists L \in \cup_{m,n} \mathcal{L}_{m,n} \text{ which is open in } \omega^*\right] \\
&\leq \sum_{n \geq 1} \sum_{m \geq 2n} (1-p)^m \#\mathcal{L}_{m,n} \leq \sum_{n \geq 1} \sum_{m \geq 2n} (4 - 4p)^m.
\end{aligned}$$

When p is close enough to 1, we have $\mathbb{P}_p[0 \not\longleftrightarrow \infty] < 1$. Thus we have $p_c \leq p < 1$. $\square$

Theorem 1.5. *For Bernoulli bond percolation on $\mathbb{Z}^2$, either there is no infinite cluster almost surely, or there exists a unique infinite cluster almost surely.*

[1] A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. Such a drawing is called a plane graph or planar embedding of the graph. The definition of the dual depends on the choice of embedding of the graph G, so it is a property of plane graphs rather than planar graphs. For planar graphs generally, there may be multiple dual graphs, depending on the choice of planar embedding of the graph.

When $\theta(p) = 0$, we have

$$\mathbb{P}_p[\exists \text{ infinite cluster}] \leq \sum_{x \in \mathbb{Z}^2} \mathbb{P}_p[x \longleftrightarrow \infty] = 0.$$

Thus there is no infinite cluster almost surely in this case. When $\theta(p) > 0$, there is positive chance to have infinite cluster. To prove Theorem 1.5, it remains two steps: First, we will show that $\mathbb{P}_p[\exists \text{ infinite cluster}] = 1$ when $\theta(p) > 0$. This part is a consequence of the ergodicity of the Bernoulli percolation—Lemma 1.6. Next, we will show that there exists a unique infinite cluster.

Let $\tau_x : \{0,1\}^{E(\mathbb{Z}^2)} \to \{0,1\}^{E(\mathbb{Z}^2)}$ be the shift by a vector $x \in \mathbb{Z}^2$ defined by

$$(\tau_x \omega)(\{a, b\}) = \omega(\{a + x, b + x\}), \quad \forall \{a, b\} \in E(\mathbb{Z}^2).$$

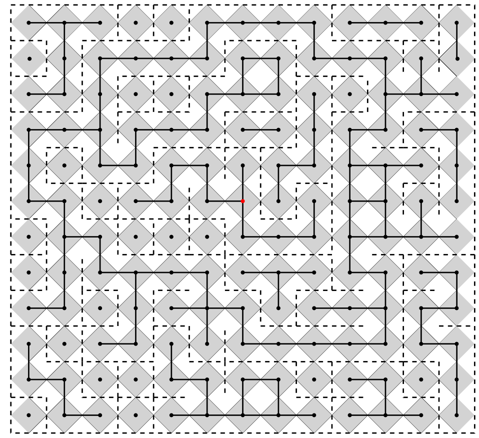


Figure 1.1: A bond configuration and its dual configuration.

For any event A, define $\tau_x A = \{\omega : \tau_x \omega \in A\}$. An event A is invariant under translation if $\tau_x A = A$ for any $x \in \mathbb{Z}^2$. A measure μ is invariant under translation if $\mu[\tau_x A] = \mu[A]$ for any event A.

Lemma 1.6 (Ergodicity of Bernoulli Percolation)**.** *Bernoulli bond percolation on $\mathbb{Z}^2$ is ergodic, i.e. any event A which is invariant under translation satisfies $\mathbb{P}_p[A] \in \{0, 1\}$.*

Proof. Let A be an event which is invariant under translations. For any $\epsilon > 0$, choose $n \geq 1$ and an event B depending on the edges in Λ_n such that $\mathbb{P}_p[A \Delta B] \leq \epsilon$. Pick $x \notin \Lambda_{2n}$. Since A is translation invariant, we have

$$\mathbb{P}_p[A] = \mathbb{P}_p[A \cap A] = \mathbb{P}_p[A \cap \tau_x A].$$

Since $\mathbb{P}_p[A \Delta B] \leq \epsilon$, we have

$$\mathbb{P}_p[A \cap \tau_x A] = \mathbb{P}_p[B \cap \tau_x B] + O(\epsilon).$$

Since B depends only on the edges in Λ_n and $x \notin \Lambda_{2n}$, we see that B and $\tau_x B$ are independent. Thus

$$\mathbb{P}_p[B \cap \tau_x B] = \mathbb{P}_p[B] \mathbb{P}_p[\tau_x B] = \mathbb{P}_p[A] \mathbb{P}_p[\tau_x A] + O(\epsilon) = \mathbb{P}_p[A]^2 + O(\epsilon).$$

Combining these, we have

$$\mathbb{P}_p[A] = \mathbb{P}_p[A]^2 + O(\epsilon).$$

This holds for any $\epsilon > 0$, thus $\mathbb{P}_p[A] = \mathbb{P}_p[A]^2$ which gives the conclusion. $\square$

Theorem 1.5 was first proved by M. Aizenman, H. Kesten and M. Newman [AKN87]. The argument presented here is due to R. M. Burton and M. Keane [BK89].

Proof of Theorem 1.5. Suppose $\theta(p) > 0$. By Lemma 1.6, we have $\mathbb{P}_p[\exists \text{ infinite cluster}] = 1$ (the event $\{\exists \text{ infinite cluster}\}$ is in the σ-algebra and it is translation invariant). It remains to show that the infinite cluster is unique. For $k \geq 1$, denote by $\mathcal{A}_k$ the event that there exist k disjoint infinite clusters; and denote by $\mathcal{A}_\infty$ the event that there exist infinitely many disjoint infinite clusters.

First, we show that $\mathbb{P}_p[\mathcal{A}_k] = 0$ for $k \geq 2$. Suppose $\mathbb{P}_p[\mathcal{A}_k] > 0$, by Lemma 1.6 we have $\mathbb{P}_p[\mathcal{A}_k] = 1$. On the event $\mathcal{A}_k$, let $\mathcal{E}_n$ be the event that all infinite clusters intersect the box Λ_n. For n large enough, we have $\mathbb{P}_p[\mathcal{E}_n] > 0$. Note that the event $\mathcal{E}_n$ does not depend on the edges inside the box Λ_n. Thus

$$\mathbb{P}_p[\mathcal{A}_1] \geq \mathbb{P}_p[\{\text{all edges in } \Lambda_n \text{ are open}\} \cap \mathcal{E}_n] > 0.$$

We have both $\mathbb{P}_p[\mathcal{A}_k] = 1$ and $\mathbb{P}_p[\mathcal{A}_1] > 0$, contradiction. This implies $\mathbb{P}_p[\mathcal{A}_k] = 0$ for $k \geq 2$.

Next, we show that $\mathbb{P}_p[\mathcal{A}_\infty] = 0$. Suppose $\mathbb{P}_p[\mathcal{A}_\infty] > 0$ and hence $\mathbb{P}_p[\mathcal{A}_\infty] = 1$. Choose n large enough such that

$$\mathbb{P}_p[\exists \text{ three infinite clusters intersecting } \Lambda_n] > 0.$$

Define T_0 to be the event that $\omega \setminus \{0\}$ has at least three distinct infinite connected components which are connected to $\{0\}$. By changing the configuration inside Λ_n, we have $\mathbb{P}_p[T_0] > 0$.

For $x \in \mathbb{Z}^2$, define $T_x = \tau_x T_0$ and we call x is a trifurcation if T_x occurs. Let $\mathcal{T}$ be the set of the trifurcations in Λ_n. On the one hand, by translation invariance, we have

$$\mathbb{E}_p[\#\mathcal{T}] = \mathbb{P}_p[T_0]\#\Lambda_n.$$

On the other hand, the number of trifurcations in Λ_n can not exceed the size of $\partial\Lambda_n$. Thus

$$\mathbb{P}_p[T_0]\#\Lambda_n \leq \#\partial\Lambda_n.$$

This is true for all n large enough and it contradicts with $\mathbb{P}_p[T_0] > 0$. This implies $\mathbb{P}_p[\mathcal{A}_\infty] = 0$. □

1.3 Subcritical: exponential decay

Theorem 1.7. *Consider Bernoulli bond percolation on $\mathbb{Z}^2$.*

- *If $p < p_c$, then there exists $c = c(p) > 0$ such that for every $n \geq 1$,*

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] \leq e^{-cn}.$$

- *If $p > p_c$, then*

$$\theta(p) = \mathbb{P}_p[0 \longleftrightarrow \infty] \geq \frac{p - p_c}{p(1 - p_c)}. \tag{1.1}$$

Theorem 1.7 was first proved by Aizenman and Barsky [AB87] and by Menshikov [Men86]. The proof presented here is due to H. Duminil-Copin and V. Tassion [DCT16]. It is based on the following crucial quantity.

Let S be a finite set of vertices containing the origin. Given such a set, we denote its edge-boundary by

$$\Delta S = \{\{x, y\} \in E(\mathbb{Z}^2) : x \in S, y \notin S\}.$$

For $p \in [0, 1]$ and $0 \in S \subset \mathbb{Z}^2$, define

$$\varphi_p(S) := p \sum_{\{x,y\} \in \Delta S} \mathbb{P}_p\left[0 \overset{S}{\longleftrightarrow} x\right].$$

Set

$$\tilde{p}_c = \sup\{p \in [0, 1] : \exists S \text{ with } \varphi_p(S) < 1\}.$$

We will prove that the two conclusions in Theorem 1.7 hold for $\tilde{p}_c$. As a consequence, we have $\tilde{p}_c = p_c$ and complete the proof of Theorem 1.7.

Proof of Theorem 1.7, Exponential decay for $p < \tilde{p}_c$. Suppose $p < \tilde{p}_c$. By definition, there exists a finite set S containing the origin such that $\varphi_p(S) < 1$. Let $N \geq 1$ such that $S \subset \Lambda_{N-1}$.

For $j \geq 2$ and assume that the event $\{0 \longleftrightarrow \partial\Lambda_{jN}\}$ holds. Define the random variable which is the cluster of the origin in S

$$\mathcal{C} := \left\{x \in S : x \overset{S}{\longleftrightarrow} 0\right\}.$$

Since $S \subset \Lambda_{N-1}$, one can find an edge $\{x, y\} \in \Delta S$ such that the following three events hold:

$$0 \overset{S}{\longleftrightarrow} x, \quad \{x, y\} \text{ is open}, \quad y \overset{\mathcal{C}^c}{\longleftrightarrow} \partial\Lambda_{jN}.$$

Using a decomposition on all the possible realizations of $\mathcal{C}$, we have

$$\begin{aligned}
\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_{jN}] &\le \sum_{\{x,y\}\in\Delta S}\sum_{C\subset S} \mathbb{P}_p\left[\{0 \xleftrightarrow{S} x\}\cap\{\mathcal{C}=C\}\cap\{\{x,y\}\text{ is open}\}\cap\{y \xleftrightarrow{C^c} \partial\Lambda_{jN}\}\right]\\
&= \sum_{\{x,y\}\in\Delta S}\sum_{C\subset S} p\mathbb{P}_p\left[\{0 \xleftrightarrow{S} x\}\cap\{\mathcal{C}=C\}\right]\mathbb{P}_p\left[\{y \xleftrightarrow{C^c} \partial\Lambda_{jN}\}\right]\\
&\le \sum_{\{x,y\}\in\Delta S}\sum_{C\subset S} p\mathbb{P}_p\left[\{0 \xleftrightarrow{S} x\}\cap\{\mathcal{C}=C\}\right]\mathbb{P}_p\left[\{0 \longleftrightarrow \partial\Lambda_{(j-1)N}\}\right]\\
&= \sum_{\{x,y\}\in\Delta S} p\mathbb{P}_p\left[\{0 \xleftrightarrow{S} x\}\right]\mathbb{P}_p\left[\{0 \longleftrightarrow \partial\Lambda_{(j-1)N}\}\right] = \varphi_p(S)\mathbb{P}_p\left[\{0 \longleftrightarrow \partial\Lambda_{(j-1)N}\}\right].
\end{aligned}$$

For the second equality, notice that the event $\{0 \xleftrightarrow{S} x\}\cap\{\mathcal{C}=C\}$ depends on the states of the edges in C while the event $\{y \xleftrightarrow{C^c} \partial\Lambda_{jN}\}$ depends on the states of the edges in C^c.

An induction on j gives

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_{jN}] \le \varphi_p(S)^{j-1},$$

as desired. ☐

For the second part of Theorem 1.7, we will need the following lemma.

Lemma 1.8. *For $p\in[0,1]$ and $n\ge 1$, we have*

$$\frac{d}{dp}\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] \ge \frac{1}{p(1-p)}\left(\inf_{S\subset\Lambda_n:0\in S}\varphi_p(S)\right)(1-\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n]).$$

Assuming Lemma 1.8 holds, and integrating the inequality between $\tilde{p}_c$ and $p>\tilde{p}_c$, we have

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] \ge \frac{p-\tilde{p}_c}{p(1-\tilde{p}_c)}.$$

Letting $n\to\infty$, we obtain the conclusion.

Proof of Lemma 1.8. Recall that an edge $e=\{x,y\}$ is pivotal for the configuration ω and the event $\{0 \longleftrightarrow \partial\Lambda_n\}$ if $\omega^e\in\{0 \longleftrightarrow \partial\Lambda_n\}$ but $\omega_e\notin\{0 \longleftrightarrow \partial\Lambda_n\}$. By Russo's formula, we have

$$\begin{aligned}
\frac{d}{dp}\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] &= \sum_{e\in\Lambda_n}\mathbb{P}_p[e\text{ is pivotal for }\{0 \longleftrightarrow \partial\Lambda_n\}]\\
&= \frac{1}{1-p}\sum_{e\in\Lambda_n}\mathbb{P}_p[e\text{ is pivotal for }\{0 \longleftrightarrow \partial\Lambda_n\},\{0 \not\longleftrightarrow \partial\Lambda_n\}].
\end{aligned}$$

Define the random subset of Λ_n:

$$\mathcal{S} := \{x\in\Lambda_n\text{ such that }x \not\longleftrightarrow \partial\Lambda_n\}.$$

The boundary of $\mathcal{S}$ corresponds to the outmost blocking circuit. When 0 is not connected to $\partial\Lambda_n$, the set $\mathcal{S}$ is always a subset of Λ_n containing the origin. By summing over all possible values for $\mathcal{S}$, we have

$$\frac{d}{dp}\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] = \frac{1}{1-p}\sum_{S\subset\Lambda_n,0\in S}\sum_{e\in\Lambda_n}\mathbb{P}_p[e\text{ is pivotal for}\{0 \longleftrightarrow \partial\Lambda_n\},\mathcal{S}=S].$$

On the event $\{\mathcal{S}=S\}$, the fact that the edge $e=\{x,y\}$ is pivotal means $0\leftrightarrow x$ in S and $e\in\Delta S$. Thus

$$\frac{d}{dp}\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] = \frac{1}{1-p}\sum_{S\subset\Lambda_n:0\in S}\sum_{\{x,y\}\in\Delta S}\mathbb{P}_p\left[0 \xleftrightarrow{S} x,\mathcal{S}=S\right].$$

The event $\{\mathcal{S} = S\}$ is measurable with respect to edges with at least one endpoint outside S and it is therefore independent of $\{0 \overset{S}{\longleftrightarrow} x\}$. Thus

$$\begin{aligned}
\frac{d}{dp}\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n] &= \frac{1}{1-p} \sum_{S\subset\Lambda_n, 0\in S} \sum_{\{x,y\}\in\Delta S} \mathbb{P}_p\left[0 \overset{S}{\longleftrightarrow} x\right] \mathbb{P}_p\left[\mathcal{S} = S\right] \\
&\geq \frac{1}{p(1-p)} \sum_{S\subset\Lambda_n, 0\in S} \varphi_p(S)\mathbb{P}_p\left[\mathcal{S} = S\right] \\
&\geq \frac{1}{p(1-p)} \inf_S \varphi_p(S) \sum_{S\subset\Lambda_n, 0\in S} \mathbb{P}_p\left[\mathcal{S} = S\right] \\
&= \frac{1}{p(1-p)} \inf_S \varphi_p(S)(1 - \mathbb{P}_p[0 \longleftrightarrow \Lambda_n]).
\end{aligned}$$

□

1.4 Computation of the critical value on $\mathbb{Z}^2$

The value 1/2 is special, since this is the self-dual point of the percolation: ω^* and ω have the same law when $p = 1/2$. This gives a prediction of the critical value. Indeed, this prediction is true.

Theorem 1.9. *For Bernoulli bond percolation on $\mathbb{Z}^2$, we have $p_c = 1/2$ and $\theta(p_c) = 0$.*

Lemma 1.10. *For Bernoulli bond percolation on $\mathbb{Z}^2$, we have $\theta(1/2) = 0$.*

Proof. In this proof, we fix $p = 1/2$, and eliminate it from the notation. We prove by contradiction. Assume $\theta(1/2) > 0$, then $\mathbb{P}[\exists \text{ infinite cluster}] = 1$. For any $\epsilon > 0$ small, we can choose n large enough such that

$$\mathbb{P}[\Lambda_n \longleftrightarrow \infty] \geq 1 - \epsilon^4.$$

Let $\mathcal{A}_L$ be the event that the left side of the box Λ_n is connected to infinity, and define $\mathcal{A}_R, \mathcal{A}_T, \mathcal{A}_B$ to the corresponding event for the right side, the top, and the bottom of Λ_n. By FKG Lemma 1.2, we have

$$\mathbb{P}[\mathcal{A}_L^c]\mathbb{P}[\mathcal{A}_R^c]\mathbb{P}[\mathcal{A}_T^c]\mathbb{P}[\mathcal{A}_B^c] \leq \mathbb{P}[\mathcal{A}_L^c \cap \mathcal{A}_R^c \cap \mathcal{A}_T^c \cap \mathcal{A}_B^c] \leq \epsilon^4.$$

By symmetry, we have $\mathbb{P}[\mathcal{A}_L^c] \leq \epsilon$. Define $\mathcal{A}_L^*, \mathcal{A}_R^*, \mathcal{A}_T^*$ and $\mathcal{A}_B^*$ for the dual configuration. Since the probabilities of $\mathcal{A}_L^c, \mathcal{A}_R^c, (\mathcal{A}_T^*)^c, (\mathcal{A}_B^*)^c$ are less than ϵ, we have

$$\mathbb{P}[\mathcal{A}_L \cap \mathcal{A}_R \cap \mathcal{A}_T^* \cap \mathcal{A}_B^*] \geq 1 - 4\epsilon > 0.$$

By changing the edges inside Λ_n, the probability to have two disjoint infinite clusters become strictly positive, contradiction. □

Proof of Theorem 1.9. We fix $p = 1/2$ and eliminate it from the notation. For a rectangle $R = [a,b]\times[c,d]$ and introduce the events

$$\mathcal{C}_h(R) := \{\{a\}\times[c,d] \overset{R}{\leftrightarrow} \{b\}\times[c,d]\}, \quad \mathcal{C}_v(R) = \{[a,b]\times\{c\} \overset{R}{\leftrightarrow} [a,b]\times\{d\}\}.$$

The events $\mathcal{C}_h^*$ and $\mathcal{C}_v^*$ are defined in terms of the dual configuration. We have the following observation:

$$\mathbb{P}[\mathcal{C}_v([0,n]^2)] + \mathbb{P}[\mathcal{C}_h^*([-1/2, n+1/2]\times[1/2, n-1/2])] = 1.$$

The rectangle $[-1/2, n+1/2]\times[1/2, n-1/2]$ is a translate of $[0, n+1]\times[0, n-1]$ which is harder to cross horizontally than $[0,n]^2$, the self-duality implies that

$$\mathbb{P}[\mathcal{C}_v([0,n]^2)] \geq 1/2.$$

Now we are ready to conclude. Lemma 1.10 implies $p_c \geq 1/2$. Assume $p_c > 1/2$, then Theorem 1.7 gives exponential decay at $p = 1/2$, which implies $\mathbb{P}[\mathcal{C}_v([0,n]^2)] \leq ne^{-cn}$ for some $c > 0$. Contradiction. □

1.5 Exercises

Exercise 1.1. *In the proof of Lemma 1.2, we argue at the beginning that, to show the conclusion, it suffices to show the conclusion for increasing functions depending only on finitely many edges. Justify this argument.*

Hint: Suppose $e_1, e_2, \ldots$ is a fixed ordering of the edges. First argue that $f_n := \mathbb{E}[f \,|\, \omega(e_1), \ldots, \omega(e_n)]$ is an increasing function, so as $g_n := \mathbb{E}[g \,|\, \omega(e_1), \ldots, \omega(e_n)]$. Next argue that $f_n \to f$ and $g_n \to g$ in L^2 and hence $\mathbb{E}[f_n g_n] \to \mathbb{E}[fg]$.

Exercise 1.2. *Consider Bernoulli bond percolation on the half-plane $\mathbb{Z} \times \mathbb{N}$. Show that $p_c(\mathbb{Z} \times \mathbb{N}) = 1/2$.*

Exercise 1.3. *A tree is a connected graph without cycle, and a rooted binary tree is a tree with a root $\emptyset$ whose degree is two and all other vertices have degree three. Denote the rooted binary tree by $T_2^{\emptyset}$. Denote by $\mathbb{P}_p$ the Bernoulli bond percolation on $T_2^{\emptyset}$ with edge-weight p. Define $\theta(p) = \mathbb{P}_p[\emptyset \longleftrightarrow \infty]$. Show that*

$$\theta(p) = \begin{cases} 0, & p \le 1/2; \\ (2p-1)/p^2, & p > 1/2. \end{cases}$$

In particular, $p_c(T_2^{\emptyset}) = 1/2$.

Exercise 1.4. *A three-regular tree $\mathcal{T}_3$ is a tree such that every vertex has degree three. We specify a vertex as the root and denote it by $\emptyset$. Denote by $\mathbb{P}_p$ the Bernoulli bond percolation on $\mathcal{T}_3$ with edge-weight p. Define $\theta(p) = \mathbb{P}_p[\emptyset \longleftrightarrow \infty]$.*

- *Show that*

  $$\theta(p) = \begin{cases} 0, & p \le 1/2; \\ 1 - ((1-p)/p)^3, & p > 1/2. \end{cases}$$

  In particular, $p_c(\mathcal{T}_3) = 1/2$.

- *When $1 > p > 1/2$, there are infinitely many infinite-clusters almost surely.*

Exercise 1.5. *Denote by $\mathbb{T}$ the triangular lattice. Denote by $\mathbb{P}_p$ the Bernoulli bond percolation on $\mathbb{T}$ with edge-weight p. Define $\theta(p) = \mathbb{P}_p[0 \longleftrightarrow \infty]$. Show that*

$$p_c(\mathbb{T}) = 2\sin(\pi/18).$$

Hint: see Figure 1.2. Conssider the bond percolation in the two graphs in Figure 1.2. Define the three events:

$$\begin{aligned} U &= \{A_1 \leftrightarrow A_2, A_2 \leftrightarrow A_3\}, \\ V &= \{A_1 \leftrightarrow A_2, A_1 \not\leftrightarrow A_3\}, \\ W &= \{A_1 \not\leftrightarrow A_2, A_2 \not\leftrightarrow A_3, A_3 \not\leftrightarrow A_1\}. \end{aligned}$$

Calculate the probabilities of U, V, W for the bond percolation on both graphs and see what is special for p that solves

$$3p - p^3 = 1.$$

Note that $p = 2\sin(\pi/18)$ solves the above equation.

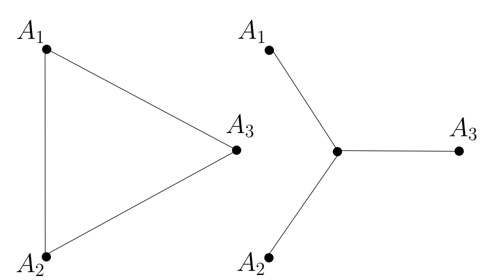


Figure 1.2

Exercise 1.6. *Theorems 1.4 to 1.7 hold for Bernoulli bond percolation in $\mathbb{Z}^d$ with $d \ge 2$.*

Exercise 1.7. *For Bernoulli bond percolation in $\mathbb{Z}^d$ with $d \geq 2$, show that*

$$\frac{1}{2d} \leq p_c(d) \leq \frac{1}{2}.$$

Conjecture 1.11. [2] *Consider Bernoulli bond percolation on $\mathbb{Z}^d$ with $d \geq 3$, we have*

$$\mathbb{P}_{p_c}[0 \longleftrightarrow \infty] = 0.$$

SRT 1. *Bernoulli percolation on higher dimension.*

Please give a report on Bernoulli percolation on higher dimension discussing the following questions.

(1) Phase transition

(2) Critical point

(3) Exponential decay

(4) Continuity of the phase transition

References: [Gri99].

SRT 2. *Bernoulli percolation on tree.*

Please give a report on Bernoulli percolation on trees discussing the following questions.

(1) Phase transition for locally finite infinite tree.

(2) Number of infinite clusters.

(3) Value of the critical point.

Reference: [Gri99, LP16]

[2] The conjecture was proved to be true for $d \geq 19$ by T. Hara and G. Slade (1990). Recently, the conjecture was proved to be true for $d \geq 11$ by R. Fitzner and R. van der Hofstad (2015).

2 Bernoulli Site Percolation on Triangular Lattice

In this section, we focus on the regular triangular lattice $\mathbb{T}$, where faces are equilateral triangles. The sites of $\mathbb{T}$ are points in $\mathbb{Z} + e^{i\pi/3}\mathbb{Z}$, and the neighboring sites are at distance one from each other. The site percolation on $\mathbb{T}$ is a probability measure $\mathbb{P}_p$ on $\omega = (\omega(x) : x \in V(\mathbb{T})) \in \{\bullet, \circ\}^{V(\mathbb{T})}$ for which each site of $V(\mathbb{T})$ is black (open) with probability p and white (closed) with probability $1-p$, independently for different sites.

Note that coloring the sites of the triangular lattice in black and white is equivalent to coloring the cells (or faces) of the honeycomb lattice as in Figure 2.1. This representation is more convenient for our eyes to detect crossings and interfaces.

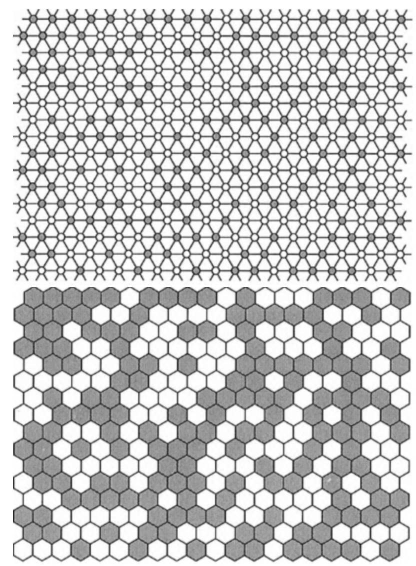


Figure 2.1

We will use the coordinate defined via the vectors 1 and $e^{i\pi/3}$. The point (n, m) will denote the point $n + me^{i\pi/3}$. Hence, $[0, n] \times [0, n]$ will denote the rhombus with side-length n. Define H_n to be the hexagon centered at the origin with graph radius n.

Note that Monotonicity, FKG Inequality and Russo's formula also hold for the site percolation. Theorems 1.4 and 1.5 also hold for the site percolation.

We will show the following conclusions for the site percolation on $\mathbb{T}$.

Theorem 2.1. *Consider Bernoulli site percolation on $\mathbb{T}$. There exists $c > 0$ such that for every $n \geq 1$,*

$$\mathbb{P}_{1/2}[0 \longleftrightarrow \partial H_n] \leq n^{-c}.$$

In particular $\theta(1/2) = 0$.

Theorem 2.2. *For Bernoulli site percolation on $\mathbb{T}$, we have $p_c = 1/2$.*

We will prove Theorem 2.1 in Section 2.1 using Russo-Seymour-Welsh. With Theorem 2.1 at hand, we can use the same proof as in Section 1.4 to obtain Theorem 2.2.

2.1 Russo-Seymour-Welsh

We focus on $p = 1/2$ in this section, since we have symmetry in this case. Since we fix $p = 1/2$ throughout this section, we eliminate it from the notation. Suppose we have the site percolation in the rhombous as in Figure 2.2, then the probability of a black left-to-right crossing is identical to the probability of a white top-to-bottom crossing. Whereas, exactly one of the two events "there exists a black left-to-right crossing" and "a white top-to-bottom crossing" occurs (due to the fact that the lattice is triangular). Therefore, the probabilities for the two events are both 1/2. This is true regardless of the size of the rhombus, and it therefore already gives information on the large-scale properties of percolation.

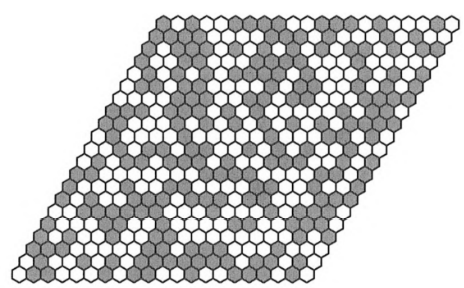


Figure 2.2

Next, we estimate the crossing probability of the site percolation inside a hexagon.

Lemma 2.3. *Denote the six sides of the boundary of the hexagon H_n by $l_1, \dots, l_6$ in counterclockwise order. Then we have*

$$\mathbb{P}[l_1 \leftrightarrow l_4] \geq 1/9.$$

Proof. By symmetry, we see that

$$\mathbb{P}[l_1 \leftrightarrow l_3 \cup l_4] = 1/2.$$

By FKG Inequality, we have

$$\mathbb{P}[l_1 \leftrightarrow l_4] \geq \mathbb{P}[\{l_1 \leftrightarrow l_3\} \cap \{l_2 \leftrightarrow l_4\}] \geq \mathbb{P}[l_1 \leftrightarrow l_3]^2.$$

Combining these two relations, we obtain the conclusion. □

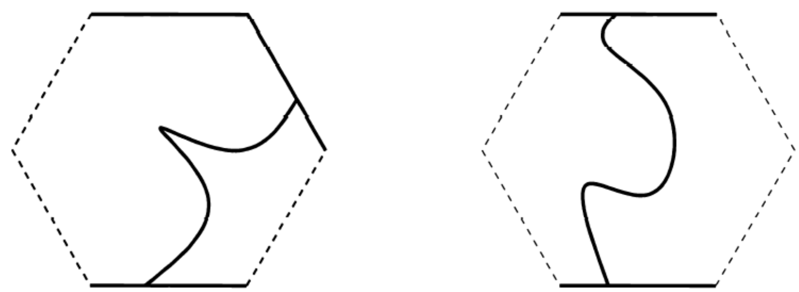


Figure 2.3: The probability for $\{l_1 \leftrightarrow l_3 \cup l_4\}$ is 1/2 by symmetry. The event $\{l_1 \leftrightarrow l_3\} \cap \{l_2 \leftrightarrow l_4\}$ implies $\{l_1 \leftrightarrow l_4\}$.

Lemma 2.4. *Let H_n' be the hexagon H_n shifted vertically such that l_1' coincides with l_4. Then we have*

$$\mathbb{P}[l_1 \leftrightarrow l_4'] \geq 1/18^2.$$

Proof. We first argue that

$$\mathbb{P}[l_1 \leftrightarrow l_2' \cup l_3' \cup l_4'] \geq 1/18. \tag{2.1}$$

Suppose there exists a crossing in H_n connecting l_1 to l_4. Starting from $l_5 \cup l_6$, one can explore site by site and find the leftmost such crossing, and denote it by $\mathcal{L}$. For any deterministic vertical crossing L, the event $\{\mathcal{L} = L\}$ is independent of the sites located to the right of L.

Assuming the existence of the path $\mathcal{L}$, let $\mathcal{L}'$ be the symmetric reflection of $\mathcal{L}$ with respect to l_4. Then $\mathcal{L}'$ is a path in H_n' connecting l_1' to l_4'. Define D to be the domain to the right of $\mathcal{L} \cup \mathcal{L}'$ inside $H_n \cup H_n'$. Inside the domain D, by symmetry, the probability of the event $\{\mathcal{L} \leftrightarrow l_2' \cup l_3' \cup l_4'\}$ is 1/2; see Figure 2.4. Therefore

$$\begin{aligned} \mathbb{P}[l_1 \leftrightarrow l_2' \cup l_3' \cup l_4'] &\geq \sum_L \mathbb{P}[\mathcal{L} = L, L \leftrightarrow l_2' \cup l_3' \cup l_4' \text{ in } D] \\ &= \sum_L \mathbb{P}[\mathcal{L} = L] \frac{1}{2} = \frac{1}{2} \mathbb{P}[l_1 \leftrightarrow l_4 \text{ in } H_n]. \end{aligned}$$

This gives (2.1). With (2.1) at hand, we find

$$\mathbb{P}[l_1 \leftrightarrow l_4'] \geq \mathbb{P}[\{l_1 \leftrightarrow l_2' \cup l_3' \cup l_4'\} \cap \{l_4' \leftrightarrow l_1 \cup l_2 \cup_3\}] \geq \mathbb{P}[l_1 \leftrightarrow l_2' \cup l_3' \cup l_4'] \mathbb{P}[l_4' \leftrightarrow l_1 \cup l_2 \cup_3] \geq 1/18^2.$$

□

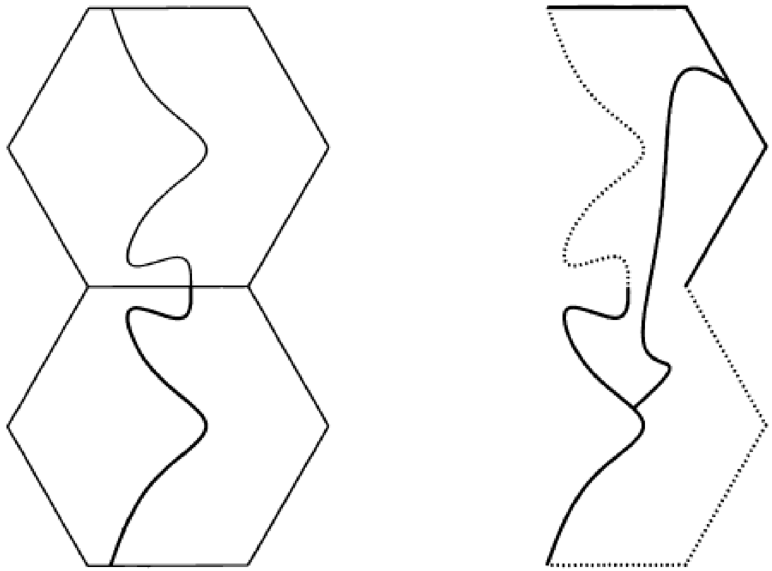


Figure 2.4

Denote by H_n^j the union of j copies of H_n aligned one by one vertically and denote by $\mathcal{C}_n^j$ the event that there exists a vertical crossing of H_n^j. By the argument in the proof of Lemma 2.4, we have

$$\mathbb{P}[\mathcal{C}_n^{2j}] \geq \mathbb{P}[\mathcal{C}_n^j]^2/4.$$

As a consequence, we have the following conclusion.

Proposition 2.5. *For all $j \geq 1$, there exists $c_j > 0$ such that*

$$\mathbb{P}[\mathcal{C}_n^j] \geq c_j, \quad \forall n.$$

Another consequence is for the existence of a black circuit in annulus, see Figure 2.5. For $m \geq 1$, denote by A_m the annulus $H_{3^m} \setminus H_{3^{m-1}}$. Then there exists $c > 0$ such that

$$\mathbb{P}[\exists \text{a black circuit in } A_m \text{ surrounding the origin}] \geq c, \quad \forall m. \tag{2.2}$$

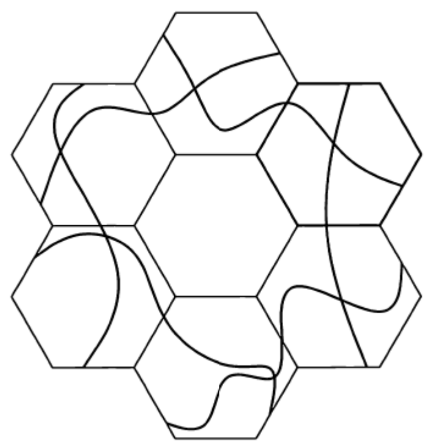


Figure 2.5

Now we can give the proof of Theorem 2.1.

Proof of Theorem 2.1. If there is a black path connecting ∂H_3 to ∂H_{3^N}, then there is no white circuit in neither annulus A_m for $2 \leq m \leq N$. Thus

$$\mathbb{P}[\partial H_3 \leftrightarrow \partial H_{3^N}] \leq p^N,$$

where $p < 1$ and it is a universal constant. This gives the conclusion. $\square$

2.2 Smirnov's Proof of Cardy's Formula

The graph

In this section, we denote by G the honeycomb lattice and by G^* its dual, i.e. the triangular lattice. Denote by G_δ and G_δ^* the lattices scaled by $\delta > 0$. The vertices of G_δ all have degree three. In the following, e will always stand for an oriented edge of G_δ, and $-e$ will stand for the same edge with opposite orientation. If e is an edge of G_δ, we will denote by e^* its dual edge, oriented in the way that the frame (e, e^*) is direct.

Each vertex $x \in G_\delta$ has three oriented edges having x as their origin. Set $\tau = e^{2\pi i/3}$. If e is one of these edges, we will denote the other two by τe and $\tau^2 e$. We have the following trivial identity: for each oriented edge e of G_δ,

$$e^* + \tau(\tau e)^* + \tau^2(\tau^2 e)^* = 0. \tag{2.3}$$

The statement

We have the following setup.

- We call $(\Omega; A, B, C, D)$ a quad if $\Omega \subset \mathbb{C}$ is a bounded simply connected domain with distinct boundary points A, B, C, D in counterclockwise order, such that $\partial\Omega$ is locally connected. We will not need the precise definition of local connectedness. Instead, we need the following property of it. Let ϕ be any conformal map from $\mathbb{U}$ onto to Ω. Then the local connectedness of $\partial\Omega$ is equivalent to that $\partial\Omega$ is a curve and is equivalent to that ϕ can be extended continuously to $\overline{\mathbb{U}}$. See [Pom92, Theorem 2.1].

- For $\delta > 0$ small, let $(\Omega_\delta; A_\delta, B_\delta, C_\delta, D_\delta)$ be an approximation of the quad $(\Omega; A, B, C, D)$ on G_δ such that $(\Omega_\delta; A_\delta, B_\delta, C_\delta, D_\delta)$ converges to $(\Omega; A, B, C, D)$ in the Carathéodory sense: there exist conformal maps ϕ_δ from $\mathbb{U}$ onto Ω_δ and conformal map ϕ from $\mathbb{U}$ onto Ω such that $\phi_\delta \to \phi$ locally uniformly as $\delta \to 0$ and $\phi^{-1}(X_\delta) \to \phi^{-1}(X)$ as $\delta \to 0$ with $X \in \{A, B, C, D\}$.

Sample the critical site percolation in Ω_δ^* and define $\mathcal{C}_\delta(\Omega; A, B, C, D)$ to be the event that there exists a black crossing in Ω_δ^* connecting the boundary arc $(A_\delta B_\delta)$ to the boundary arc $(C_\delta D_\delta)$.

Theorem 2.6. *[Smi01] For the critical Bernoulli site percolation on $\mathbb{T}$, we have the followings.*

- *The probability of $\mathcal{C}_\delta$ is convergent:*

$$\mathbb{P}[\mathcal{C}_\delta(\Omega; A, B, C, D)] \to f(\Omega; A, B, C, D), \quad \text{as } \delta \to 0.$$

- *The function f is conformally invariant: for any conformal map ϕ on Ω, we have*

$$f(\phi(\Omega); \phi(A), \phi(B), \phi(C), \phi(D)) = f(\Omega; A, B, C, D).$$

- *When Ω equals the equilateral triangle with three vertices A, B, C (note that the point D is located on the segment CA), we have*

$$f(\Omega; A, B, C, D) = \frac{|CD|}{|CA|}.$$

For $z \in \Omega$, suppose z_δ is an approximation of z on G_δ. Define $E_A^\delta(z)$ to be the event that there exists a black path in Ω_δ^* separating the points A_δ, z_δ from the points B_δ, C_δ. Set $H_A^\delta(z) = \mathbb{P}[E_A^\delta(z)]$. Define $E_B^\delta(z), E_C^\delta(z)$ and $H_B^\delta(z), H_C^\delta(z)$ similarly. Note that $\mathbb{P}[\mathcal{C}_\delta(\Omega; A, B, C, D)] = H_A^\delta(D)$. The proof of Theorem 2.6 consists of three steps.

- First Step: Tightness. We first argue that the family $\{H_A^\delta, H_B^\delta, H_C^\delta\}_{\delta>0}$ is tight under the topology of uniform convergence.

- Second Step: Holomorphicity. Define

$$H^\delta(z) = H_A^\delta(z) + \tau H_B^\delta(z) + \tau^2 H_C^\delta(z), \quad S^\delta(z) = H_A^\delta(z) + H_B^\delta(z) + H_C^\delta(z).$$

  Since $\{H_A^\delta, H_B^\delta, H_C^\delta\}_{\delta>0}$ is tight, the functions $\{H^\delta, S^\delta\}_{\delta>0}$ have subsequential limits. Suppose (H, S) is a subsequential limit, we will argue that H and S are holomorphic.

- Third Step: Boundary Value. In the previous step, we see that the functions H and S are holomorphic and bounded, and hence they are uniquely determined by their boundary values. For the function S, since it is holomorphic and real-valued, it has to be a constant. Since $S(A) = 1$, we find $S \equiv 1$. For the function H, we easily find $H(A) = 1, H(B) = \tau, H(C) = \tau^2$. Consider its value on the arc (BC). For $z \in (BC)$, we find $H_A(z) = 0$ and $H_B(z) + H_C(z) = 1$. Along the arc $[BC]$, the function H is one-to-one and induces a continuous map from $[BC]$ onto $[\tau, \tau^2]$. Similar conclusion holds for the arc $[CA]$ and $[AB]$. Note that H is holomorphic in Ω and that H extends continuously to $\overline{\Omega}$ and induces a continuous bijection from $\partial\Omega$ to $\partial\triangleright$. By Principle of Corresponding Boundaries, we conclude that H is the conformal map from Ω onto $\triangleright$ which sends (A, B, C) to $(1, \tau, \tau^2)$.

Now we can conclude the proof of Theorem 2.6: for any subsequential limit $(H_A^\delta, H_B^\delta, H_C^\delta) \to (H_A, H_B, H_C)$, we have

$$H_A + H_B + H_C = 1, \quad H_A + \tau H_B + \tau^2 H_C = H,$$

where H is the conformal map from Ω onto $\triangleright$ which sends (A, B, C) to $(1, \tau, \tau^2)$. Thus

$$H_A(z) = \frac{2\Re(H(z)) + 1}{3},$$

which completes the proof of Theorem 2.6.

Tightness

For two points $z, w \in \Omega$, consider the events $E_A^\delta(z)$ and $E_A^\delta(w)$. Note that the difference $|H_A^\delta(z) - H_A^\delta(w)|$ is bounded by $\mathbb{P}[E_A^\delta(z) \setminus E_A^\delta(w)] + \mathbb{P}[E_A^\delta(w) \setminus E_A^\delta(z)]$. Suppose $|z - w| \le 1/100$ and $\mathrm{dist}((z+w)/2, \partial\Omega) \ge 1$, and denote by $\mathcal{A}$ the annulus with center $(z + w)/2$, inradius $|z - w|$ and outradius 1. If there exists a black circuit in the annulus $\mathcal{A}$ surrounding the center, then $E_A^\delta(z)$ and $E_Z^\delta(w)$ hold simultaneously or fail simultaneously. Thus $\mathbb{P}[E_A^\delta(z) \setminus E_A^\delta(w)]$ and $\mathbb{P}[E_A^\delta(w) \setminus E_A^\delta(z)]$ are bounded by the probability that there is no black circuit in the annulus $\mathcal{A}$. By RSW, we obtain the following: there exists two positive constants K and $\alpha > 0$ such that

$$|H_A^\delta(z) - H_A^\delta(w)| \le K|z - w|^\alpha;$$

and a similar bound for H_B^δ and H_C^δ. Hence, if we suitably extend these functions continuously to Ω, we obtain a family of uniformly Hölder maps from Ω to $\mathbb{R}$. The family is then relatively compact with respect to uniform convergence.

Holomorphicity

We first recall some standard facts in complex analysis. A function $f : \Omega \to \mathbb{C}$ is holomorphic if for any $z \in \Omega$, the complex derivative exists: $f'(z) = \lim_{|\epsilon| \to 0}(f(z + \epsilon) - f(z))/\epsilon$. By Morera's theorem, f is holomorphic if and only if for every simple closed smooth curve γ in Ω, we have $\oint_\gamma f = 0$. We will use Morera's theorem to show that the functions H and S are holomorphic. We only show the conclusion for H and the holomorhpicity of S can be derived similarly.

Suppose γ is a simple closed smooth curve in Ω and let γ_δ be an approximation of γ in Ω_δ, i.e. a finite chain $(\gamma_\delta(k), 0 \le k \le N_\delta)$ such that $\gamma_\delta(k+1)$ and $\gamma_\delta(k)$ are nearest neighbours, the Hausdorff distance between γ_δ and γ goes to zero as $\delta \to 0$, and that N_δ is of order $1/\delta$.

Suppose H is a subsequential limit of H^δ. Then we have

$$I^\delta(\gamma) := \sum_{k=0}^{N_\delta} \frac{1}{2}(H^\delta(\gamma_\delta(k)) + H^\delta(\gamma_\delta(k+1)))(\gamma_\delta(k+1) - \gamma_\delta(k)) \to \oint_\gamma H(z)dz.$$

To show the holomorphicity of H, it is sufficient to show $I^\delta(\gamma) \to 0$ as $\delta \to 0$. For any edge e pointing from x to y, define

$$H^\delta(e) = \frac{1}{2}(H^\delta(x) + H^\delta(y)), \quad \partial_e H^\delta = H^\delta(y) - H^\delta(x).$$

Then we have

$$I^\delta(\gamma) = \sum_{e \in \gamma_\delta} e H^\delta(e) = \sum_{f \in \mathring{\gamma}_\delta} \sum_{e \in \partial f} e H^\delta(e),$$

where the first summation $e \in \gamma_\delta$ is summing over all edges along γ_δ in counterclockwise order, the second summation $f \in \mathring{\gamma}_\delta$ is summing over all faces surrounded by γ_δ, and the third summation $e \in \partial f$ is summing over all edges along ∂f in counterclockwise order.

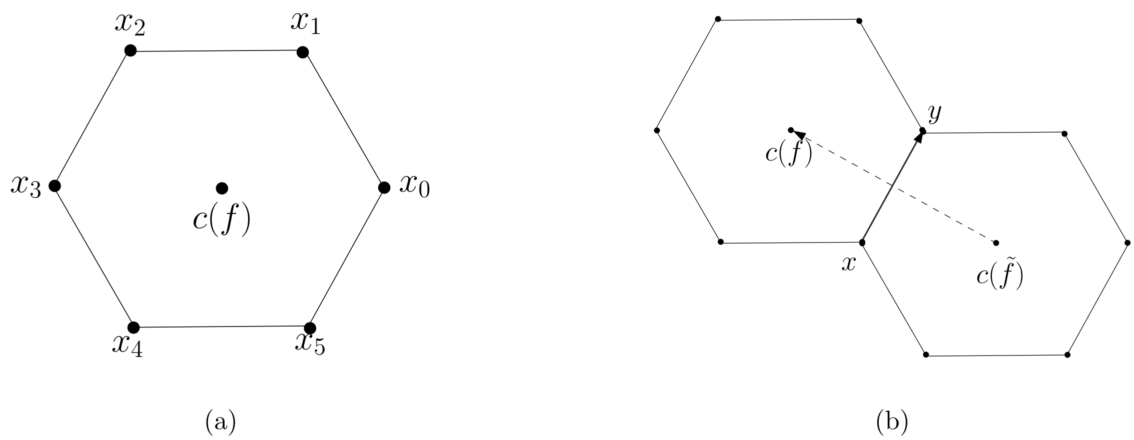


Figure 2.6

For each face f, we have "integrate by parts", see Figure 2.6(a):

$$\begin{aligned}
\sum_{e \in \partial f} e H^\delta(e) &= \sum_0^5 (x_{k+1} - x_k) \frac{H^\delta(x_{k+1}) + H^\delta(x_k)}{2} \\
&= \sum_0^5 (x_{k+1} - c(f)) \frac{H^\delta(x_{k+1}) + H^\delta(x_k)}{2} - \sum_0^5 (x_k - c(f)) \frac{H^\delta(x_{k+1}) + H^\delta(x_k)}{2} \\
&= -\sum_0^5 (x_k - c(f)) \frac{H^\delta(x_{k+1}) - H^\delta(x_{k-1})}{2} \\
&= -\sum_0^5 \left(\frac{x_k + x_{k+1}}{2} - c(f) \right) (H^\delta(x_{k+1}) - H^\delta(x_k)).
\end{aligned}$$

Putting this back in the summation of $I^\delta(\gamma)$, for any edge $e = (x, y)$ in the interior of the region surrounded by γ_δ, it is shared by two faces f and $\tilde{f}$, see Figure 2.6(b), we have

$$\begin{aligned}
&\left(\frac{x+y}{2} - c(f) \right) (H^\delta(y) - H^\delta(x)) + \left(\frac{x+y}{2} - c(\tilde{f}) \right) (H^\delta(x) - H^\delta(y)) \\
&= (c(\tilde{f}) - c(f))(H^\delta(y) - H^\delta(x)) \\
&= -e^* \partial_e H^\delta.
\end{aligned}$$

For any edge $e = (x, y)$ along γ_δ, by the argument in "Tightness", we find

$$|H^\delta(y) - H^\delta(x)| \le 3K\delta^\alpha.$$

Combining with the fact that N_δ is of order $1/\delta$, we have

$$I^\delta(\gamma) = \frac{1}{2}\sum_{e\subset\gamma_\delta} e^*\partial_e H^\delta + o(1),$$

where the summation $e \subset \gamma_\delta$ is summing over all oriented edges e lying in the inside of γ_δ. We will show that

$$\sum_{e\subset\gamma_\delta} e^*\partial_e H^\delta = 0. \tag{2.4}$$

Note that (2.4) implies $I^\delta(\gamma) \to 0$ and hence completes the proof.

We will eliminate δ from the notation in the following. For an edge $e = (x, y)$ lying in the inside of γ_δ, define $P_A(e) = \mathbb{P}[E_A(y) \setminus E_A(x)]$, then

$$\partial_e H_A = H_A(y) - H_A(x) = P_A(e) - P_A(-e).$$

Thus

$$\sum_{e\subset\gamma_\delta} e^*\partial_e H^\delta = 2\sum_{e\subset\gamma_\delta} e^*(P_A(e) + \tau P_B(e) + \tau^2 P_C(e)).$$

Combining with Lemma 2.7,

Lemma 2.7. *[Color Switching] For any edge e lying in the inside of γ_δ,*

$$P_A(e) = P_B(\tau e) = P_C(\tau^2 e).$$

We see that

$$\begin{aligned}\sum_{e\subset\gamma_\delta} e^*\partial_e H^\delta &= 2\sum_{e\subset\gamma_\delta} e^*(P_A(e) + \tau P_B(e) + \tau^2 P_C(e)) \\ &= 2\sum_{e\subset\gamma_\delta} e^*(P_A(e) + \tau P_A(\tau^2 e) + \tau^2 P_A(\tau e)) \\ &= 2\sum_{e\subset\gamma_\delta} \left(e^* + \tau(\tau e)^* + \tau^2(\tau^2 e)^*\right) P_A(e) = 0. \qquad \text{(By (2.3))}\end{aligned}$$

This completes the proof of (2.4) and completes the proof. It remains to show Lemma 2.7. We prove it by drawing figures, see Figure 2.7.

As a consequence of Theorem 2.6, we can derive the convergence of the interface in the critical site percolation Theorem 2.8.

Theorem 2.8. *Consider Bernoulli site percolation on triangular lattice. The interface converges weakly to* SLE(6)*. In particular, we have*

$$\mathbb{P}_{p_c}[0 \longleftrightarrow \partial\Lambda_n] = n^{-5/48+o(1)}, \quad \textit{as } n \to \infty.$$

$$\theta(p) = (p - p_c)^{5/36+o(1)}, \quad \textit{as } p \downarrow p_c.$$

Proof. [LSW02, SW01, Nol08]. □

These conclusions are believed to be true for the bond percolation, but they are still open.

Conjecture 2.9. *Theorem 2.8 holds for the critical Bernoulli bond percolation on $\mathbb{Z}^2$.*

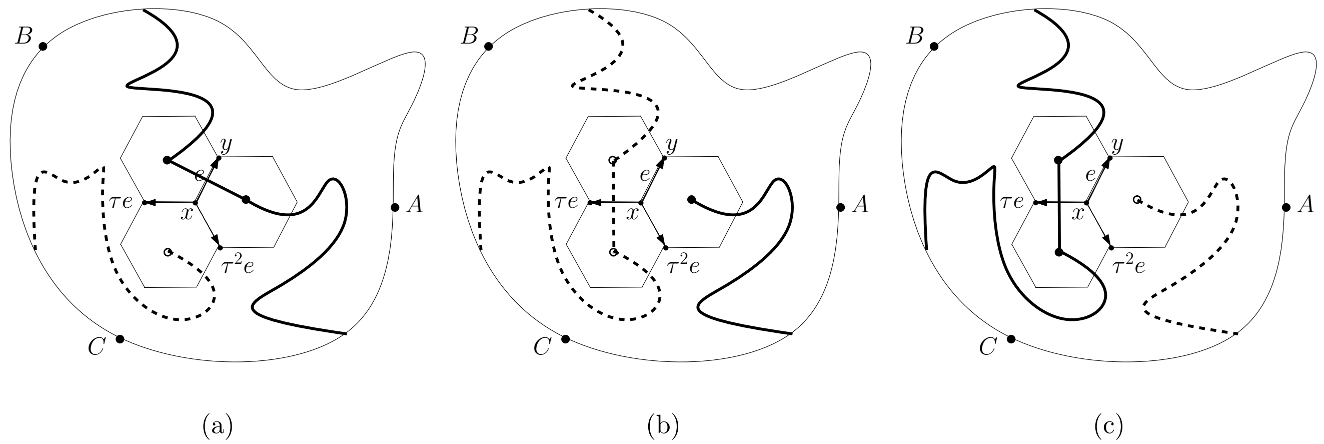


Figure 2.7: Suppose $e = (x, y)$, then $P_A(e)$ is the probability of $E_A(y) \setminus E_A(x)$. The event $E_A(y) \setminus E_A(x)$ means there exists a black path separating A, y from B, C; while there exists a white path connecting x to the arc (BC), as in (a). We explore the percolation site by site starting from C and suppose we find the rightmost white path connecting x to (BC) and the leftmost black path connecting y to (CA). The sites in the remaining domain are independent from what we explored so far. We switch the colors of the sites in the remaining domain, as in (b). Then we switch the color of all sites, as in (c). These operations explain that $P_A(e)$ equals $P_B(\tau e)$.

2.3 Exercises

Exercise 2.1. *Consider Bernoulli site percolation on $\mathbb{Z}^2$. Show that $p_c > 1/2$.*

SRT 3. *Bernoulli site percolation vs. bond percolation*

Please give a report on Bernoulli site percolation discussing the following questions.

(1) Relation between $p_c^s(G)$ and $p_c^b(G)$.

(2) $p_c^s(\mathbb{Z}^2)$.

References: [Gri99, BR06, GM13].

SRT 4. *Bernoulli site percolation: the critical arm exponents and the scaling relations.*

Please give a report on Bernoulli site percolation discussing the following topics.

(1) convergence of the interface to SLE(6)

(2) polychromatic arm exponents (boundary and interior)

(3) one-arm exponent

(4) monochromatic arm exponents

(5) near-critical percolation, characteristic length

References: [Smi01, SW01, Wer04, Nol08].

3 Random Cluster Model

The random-cluster model, also called Fortuin-Kasteleyn percolation, is a dependent percolation model on $\mathbb{Z}^d$ which is intimately related to the Potts model.

Let G be a finite subgraph of $\mathbb{Z}^d$. Boundary conditions ξ are given by a partition $P_1 \sqcup \cdots \sqcup P_k$ of $\partial G := \{x \in G : \exists y \notin G, \{x,y\} \in E(\mathbb{Z}^d)\}$. Two vertices are wired in ξ if they belong to the same P_j. The graph obtained from the configuration ω by identifying the wired vertices together is denoted by ω^ξ. Boundary conditions should be understood informally as encoding how sites are connected outside of G. Let $o(\omega)$ and $c(\omega)$ denote the number of open and closed edges of ω and $k(\omega^\xi)$ the number of connected components of the graph ω^ξ. The probability measure $\phi^\xi_{p,q,G}$ of the random-cluster model on G with edge-weight $p \in [0,1]$, cluster-weight $q > 0$ and boundary conditions ξ is defined by

$$\phi^\xi_{p,q,G}[\omega] = \frac{1}{\mathcal{Z}^\xi_{p,q,G}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega^\xi)},$$

for every configuration $\omega \in \{0,1\}^{E(G)}$. The constant $\mathcal{Z}^\xi_{p,q,G}$ is a normalizing constant, referred to as the partition function. For $q = 1$, the random-cluster model is Bernoulli bond percolation.

We list a few boundary conditions that will come back later in the notes.

- Wired boundary conditions. They are specified by the fact that all the vertices on the boundary are pairwise wired (the partition is equal to $\{\partial G\}$). The random-cluster measure with wired boundary conditions on G is denoted by $\phi^1_{p,q,G}$.
- Free boundary conditions. They are specified by no wiring between vertices on the boundary (the partition is composed of singletons only). The random-cluster measure with free boundary conditions on G is denoted by $\phi^0_{p,q,G}$.
- Dobrushin boundary conditions. Let $(\Omega; a, b)$ be a discrete simply connected domain Ω with two vertices a and b on its boundary $\partial\Omega$. We call such a triplet a Dobrushin domain. Since Ω is simply connected, $\partial\Omega$ is separated into two boundary arcs denoted by (ab) (the arc from a to b counterclockwise) and (ba) (the arc from b to a counterclockwise). The Dobrushin boundary conditions are defined to be free on (ab) and wired on (ba). The corresponding measure will be denoted by $\phi^{a,b}_{p,q,G}$.
- Boundary conditions induced by a configuration outside G. For a configuration ξ on $E(\mathbb{Z}^2) \setminus E(G)$, the boundary conditions induced by ξ are defined by the partition $P_1 \sqcup \cdots \sqcup P_k$. where x and y are in the same P_j if and only if there exists an open path in ξ connecting x and y. We identify the boundary conditions induced by ξ with the configuration itself, and we denote the random-cluster measure with these boundary conditions by $\phi^\xi_{p,q,G}$.

The random-cluster model satisfies the domain Markov property. Suppose that $G' \subset G$ are two finite subgraphs of $\mathbb{Z}^2$. Fix $p \in [0,1], q > 0$ and ξ some boundary conditions on ∂G. Let X be a random variable which is measurable with respect to edges in $E(G')$. Then we have

$$\phi^\xi_{p,q,G}\left[X \,|\, \omega_e = \psi_e, \forall e \in E(G) \setminus E(G')\right] = \phi^{\psi^\xi}_{p,q,G'}[X], \quad \forall \psi \in \{0,1\}^{E(G)\setminus E(G')},$$

where ψ^ξ is the partition on $\partial G'$ obtained as follows: two vertices $x, y \in \partial G'$ are wired if they are connected in ψ^ξ.

3.1 FKG Inequality

Theorem 3.1 (FKG Inequality)**.** *Fix $p \in [0,1]$, $q \geq 1$, a finite graph G and some boundary conditions ξ. For any two increasing events A and B, we have*

$$\phi^\xi_{p,q,G}[A \cap B] \geq \phi^\xi_{p,q,G}[A]\phi^\xi_{p,q,G}[B].$$

In order to prove FKG inequality, we will use Holley criterion. Given two probability measures μ_1, μ_2, we write $\mu_1 \leq_{st} \mu_2$, and we say μ_1 is stochastically smaller than μ_2, if $\mu_1[A] \leq \mu_2[A]$ for all increasing event A. We call a probability measure μ strictly positive if $\mu[\omega] > 0$ for all configurations ω.

Theorem 3.2 (Holley inequality)**.** *Let μ_1, μ_2 be strictly positive probability measures on the finite state space such that*

$$\mu_2[\omega^e]\mu_1[\eta_e] \geq \mu_2[\omega_e]\mu_1[\eta^e], \quad \forall e \in E, \forall \eta \leq \omega, . \tag{3.1}$$

Then $\mu_1 \leq_{st} \mu_2$.

Proof. We will show that, under Condition (3.1), the two probability measures μ_1 and μ_2 can be coupled on the space $\{(\eta, \omega) : \eta \leq \omega\}$, then the conclusion follows immediately. This is achieved by constructing a certain Markov chain with the coupled measure as invariant measure.

Here is a preliminary calculation. Let μ be a strictly positive probability measure. We may construct a reversible Markov chain with invariant measure μ by choosing a suitable generator satisfying the detailed balance equations. Let Q be given by

$$Q(\omega_e, \omega^e) = 1, \quad Q(\omega^e, \omega_e) = \frac{\mu[\omega_e]}{\mu[\omega^e]}, \quad \forall e.$$

We set $Q(\omega, \omega') = 0$ for all other pairs ω, ω' with $\omega \neq \omega'$. The diagonal elements $Q(\omega, \omega)$ are chosen in such a way that

$$\sum_{\omega'} Q(\omega, \omega') = 0, \quad \forall \omega.$$

Then we find

$$\mu(\omega)Q(\omega, \omega') = \mu(\omega')Q(\omega', \omega), \quad \forall \omega, \omega',$$

and therefore Q generates a Markov chain on the state space which is reversible with respect to μ. Since the Markov chain is irreducible, its stationary measure is unique which is μ.

We follow a similar route for pairs of configurations. Let μ_1 and μ_2 satisfy the hypotheses of the theorem, and let $S = \{(\eta, \omega) : \eta \leq \omega\}$. Define H on $S \times S$ by

$$\begin{aligned} H(\eta_e, \omega; \eta^e, \omega^e) &= 1, \\ H(\eta, \omega^e; \eta_e, \omega_e) &= \frac{\mu_2[\omega_e]}{\mu_2[\omega^e]}, \\ H(\eta^e, \omega^e; \eta_e, \omega^e) &= \frac{\mu_1[\eta_e]}{\mu_1[\eta^e]} - \frac{\mu_2[\omega_e]}{\mu_2[\omega^e]} \geq 0, \end{aligned} \tag{by (3.1)}$$

for all $(\eta, \omega) \in S$ and $e \in E$; all other off-diagonal values of H are set to be zero; and the diagonal terms $H(\eta, \omega; \eta, \omega)$ are chosen such that

$$\sum_{(\eta', \omega')} H(\eta, \omega; \eta', \omega') = 0 \quad \forall (\eta, \omega) \in S.$$

Let (X_t, Y_t) be a Markov chain on S with generator H, and set $X_0 = 0$ (the state of all zeros) and $Y_0 = 1$ (the state of all ones). Since all transitions retain the ordering of the two components, we have $\mathbb{P}[X_t \leq Y_t, \forall t] = 1$.

We first check that the stationary measure for Y is μ_2. It is clear that the generator of (Y_t) is given by

$$Q_Y(\omega_e; \omega^e) = 1, \quad Q_Y(\omega^e; \omega_e) = \frac{\mu_2[\omega_e]}{\mu_2[\omega^e]}.$$

Thus the chain (Y_t) is reversible with respect to μ_2. Hence μ_2 is the stationary measure.

Next, we check that the stationary measure for X is μ_1. Let us calculate the generator of X_t:

$$\begin{aligned}
\mathbb{P}[X_{t+h}=\eta^e \mid X_t=\eta_e] &= \sum_{\omega} \mathbb{P}[X_{t+h}=\eta^e \mid (X_t,Y_t)=(\eta_e,\omega)]\mathbb{P}[(X_t,Y_t)=(\eta_e,\omega) \mid X_t=\eta_e] \\
&= \sum_{\omega} (h+o(h))\mathbb{P}[(X_t,Y_t)=(\eta_e,\omega) \mid X_t=\eta_e] = h+o(h). \\
\mathbb{P}[X_{t+h}=\eta_e \mid X_t=\eta^e] &= \sum_{\omega:\omega(e)=1} \mathbb{P}[(X_{t+h},Y_{t+h})=(\eta_e,\omega_e) \mid (X_t,Y_t)=(\eta^e,\omega^e)]\mathbb{P}[Y_t=\omega^e \mid X_t=\eta^e] \\
&+ \sum_{\omega:\omega(e)=1} \mathbb{P}[(X_{t+h},Y_{t+h})=(\eta_e,\omega^e) \mid (X_t,Y_t)=(\eta^e,\omega^e)]\mathbb{P}[Y_t=\omega^e \mid X_t=\eta^e] \\
&= \sum_{\omega:\omega(e)=1} \left(\frac{\mu_2[\omega_e]}{\mu_2[\omega^e]}h+o(h)\right)\mathbb{P}[Y_t=\omega^e \mid X_t=\eta^e] \\
&+ \sum_{\omega:\omega(e)=1} \left(\left(\frac{\mu_1[\eta_e]}{\mu_2[\eta^e]}-\frac{\mu_2[\omega_e]}{\mu_2[\omega^e]}\right)h+o(h)\right)\mathbb{P}[Y_t=\omega^e \mid X_t=\eta^e] \\
&= \frac{\mu_1[\eta_e]}{\mu_1[\eta^e]}h+o(h).
\end{aligned}$$

Therefore, the generator of (X_t) is given by

$$Q_X(\eta_e;\eta^e)=1, \quad Q_X(\eta^e;\eta_e)=\frac{\mu_1[\eta_e]}{\mu_1[\eta^e]}.$$

Thus (X_t) is reversible with respect to μ_1, and μ_1 is the stationary measure.

Finally, let ν be an invariant measure for the Markov chain (X_t,Y_t). Since X (resp. Y) has the unique stationary measure μ_1 (resp. μ_2), the marginals of ν are μ_1 and μ_2. Thus ν is the desired coupling of μ_1 and μ_2. □

Proof of Theorem 3.1. Denote $\phi^{\xi}_{p,q,G}$ by μ and set $\mu_1=\mu$ and $\mu_2[\cdot]=\mu[\cdot \mid B]$. By Holley inequality, to show the conclusion, we only need to check that μ_1 and μ_2 satisfy Condition (3.1). To this end, it is sufficient to check, for $\eta \le \omega$,

$$\mathbb{1}_{\{\omega^e\in B\}}\,\mu[\omega^e]\mu[\eta_e] \ge \mathbb{1}_{\{\omega_e\in B\}}\,\mu[\omega_e]\mu[\eta^e]. \tag{3.2}$$

Since B is increasing, the event $\{\omega_e \in B\}$ implies $\{\omega^e \in B\}$. We may assume $\omega_e \in B$ in the following. Note that, up to the normalization constant $\mathcal{Z}^{\xi}_{p,q,G}$, we have

$$\begin{aligned}
\mu[\omega^e]\mu[\eta_e] &= p^{o(\omega^e)+o(\eta_e)}(1-p)^{c(\omega^e)+c(\eta_e)}q^{k^{\xi}(\omega^e)+k^{\xi}(\eta_e)}, \\
\mu[\omega_e]\mu[\eta^e] &= p^{o(\omega_e)+o(\eta^e)}(1-p)^{c(\omega_e)+c(\eta^e)}q^{k^{\xi}(\omega_e)+k^{\xi}(\eta^e)}.
\end{aligned}$$

Since $q \ge 1$, it is sufficient to show

$$k^{\xi}(\omega^e)+k^{\xi}(\eta_e) \ge k^{\xi}(\omega_e)+k^{\xi}(\eta^e).$$

This is guaranteed by the fact that $\eta \le \omega$. □

Corollary 3.3 (Monotonicity). *Fix $p \le p'$ and $q \ge 1$, a finite graph G and some boundary conditions ξ. For any increasing event A, we have*

$$\phi^{\xi}_{p,q,G}[A] \le \phi^{\xi}_{p',q,G}[A].$$

Proof. Note that

$$\phi^{\xi}_{p',q,G}[A] = \frac{\phi^{\xi}_{p,q,G}[Y\,\mathbb{1}_A]}{\phi^{\xi}_{p,q,G}[Y]}, \quad \text{where } Y(\omega) := \left(\frac{p'(1-p)}{p(1-p')}\right)^{o(\omega)}.$$

Since $p \le p'$, the function Y is increasing. Thus by FKG, we have

$$\phi^{\xi}_{p',q,G}[A] = \frac{\phi^{\xi}_{p,q,G}[Y\,\mathbb{1}_A]}{\phi^{\xi}_{p,q,G}[Y]} \ge \phi^{\xi}_{p,q,G}[A],$$

as desired. □

Corollary 3.4 (Comparison between boundary conditions)**.** *Fix $p \in [0,1]$ and $q \ge 1$, a finite graph G. For any boundary conditions $\xi \le \psi$ (meaning that two vertices wired in ξ are also wired in ψ), we have, for any increasing event A,*

$$\phi^{\xi}_{p,q,G}[A] \le \phi^{\psi}_{p,q,G}[A].$$

In particular, for any boundary conditions ξ, and for any increasing event A, we have

$$\phi^{0}_{p,q,G}[A] \le \phi^{\xi}_{p,q,G}[A] \le \phi^{1}_{p,q,G}[A].$$

Another consequence of FKG is the so-called finite-energy property: for any $q \ge 1$, any finite graph G, any boundary conditions ξ, and any $\psi \in \{0,1\}^{E(G)}$, we have

$$\frac{p}{p+(1-p)q} \le \phi^{\xi}_{p,q,G}\left[\omega(f)=1 \,|\, \omega(e)=\psi(e)\ \forall e \in E(G)\setminus\{f\}\right] \le p. \tag{3.3}$$

3.2 Infinite-volume measure

The definition of an infinite volume random-cluster measure is not direct. Indeed, one cannot count the number of open or closed edges in $\mathbb{Z}^2$ since they could be infinite. We will obtain infinite-volume measure by taking a sequence of measures on larger and larger boxes Λ_n.

Let ξ_n be a sequence of boundary conditions. The sequence $\phi^{\xi_n}_{p,q,\Lambda_n}$ is said to converge to the infinite-volume measure $\phi_{p,q}$ if

$$\lim_n \phi^{\xi_n}_{p,q,\Lambda_n}[A] = \phi_{p,q}[A],$$

for any event A depending only on the status of finitely many edges.

Proposition 3.5. *Fix $p \in [0,1]$ and $q \ge 1$. There exists two (possibly equal) infinite-volume random-cluster measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$, called the infinite-volume random-cluster measures with free and wired boundary conditions respectively, such that for any event A depending on a finite number of edges,*

$$\lim_{n\to\infty} \phi^{1}_{p,q,\Lambda_n}[A] = \phi^1_{p,q}[A], \quad \lim_{n\to\infty} \phi^{0}_{p,q,\Lambda_n}[A] = \phi^0_{p,q}[A].$$

Proof. We deal with the free boundary conditions, the wired boundary conditions can be treated similarly. Fix an increasing event A depending on edges in Λ_N only. Applying the domain Markov property and the comparison between boundary conditions, we see,for any $n \ge N$,

$$\phi^{0}_{p,q,\Lambda_{n+1}}[A] \ge \phi^{0}_{p,q,\Lambda_n}[A].$$

The increasing sequence $(\phi^{0}_{p,q,\Lambda_n}[A])_{n\ge N}$ converges to a limit, denoted by $P[A]$, as $n \to \infty$.

Since any event B depending on finitely many edges can be written by inclusion-exclusion as a combination of increasing events, taking the combination of the ϕ^{0}_{p,q,Λ_n}-probability defines a natural value $P[B]$ for which

$$\phi^{0}_{p,q,\Lambda_n}[B] \to P[B], \quad \text{as } n \to \infty.$$

The fact that $(\phi^{0}_{p,q,\Lambda_n})_{n\ge 0}$ are probability measures implies that the function P (which is defined on the set of events depending on finitely many edges) can be extended into a probability measure on $\mathbb{Z}^2$. We denote this measure by $\phi^0_{p,q}$. □

Lemma 3.6. *Fix $p \in [0,1]$ and $q \geq 1$. The infinite-volume measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$ are translation invariant and are ergodic.*

Proof. We fix the parameters p, q and hence eliminate them from the notation. We first show that ϕ^1 is translation invariant and the proof for ϕ^0 is similar. To this end, we only need to show that $\phi^1[A] = \phi^1[\tau_x A]$ for any increasing event A depending only on finitely many edges and $x \in \mathbb{Z}^2$ which is a neighbor of the origin. We have

$$\phi^1[A] = \lim_n \phi^1_{\Lambda_n}[A] = \lim_n \phi^1_{\tau_x \Lambda_n}[\tau_x A].$$

Note that $\Lambda_{n-1} \subset \tau_x \Lambda_n \subset \Lambda_{n+1}$. By the domain Markov property and the comparison between boundary conditions, we have

$$\phi^1_{\Lambda_{n+1}}[\tau_x A] \leq \phi^1_{\tau_x \Lambda_n}[\tau_x A] \leq \phi^1_{\Lambda_{n-1}}[\tau_x A].$$

This implies that

$$\lim_n \phi^1_{\tau_x \Lambda_n}[\tau_x A] = \phi^1[\tau_x A].$$

Thus $\phi^1[A] = \phi^1[\tau_x A]$.

Next, we show that ϕ^1 is ergodic. We can repeat the same proof of Lemma 1.6 as long as we prove the following fact: for any increasing events A and B depending on finitely many edges,

$$\lim_{|x| \to \infty} \phi^1[A \cap \tau_x B] = \phi^1[A]\phi^1[B]. \tag{3.4}$$

Suppose the events A and B depend only on edges in Λ_N. The FKG inequality implies that

$$\phi^1[A \cap \tau_x B] \geq \phi^1[A]\phi^1[\tau_x B] = \phi^1[A]\phi^1[B].$$

For the other direction, the comparison between boundary conditions implies that for $|x| \geq 2N$,

$$\phi^1[A \cap \tau_x B] \leq \phi^1_{\Lambda_{|x|/2}}[A]\phi^1_{\tau_x \Lambda_{|x|/2}}[\tau_x B] = \phi^1_{\Lambda_{|x|/2}}[A]\phi^1_{\Lambda_{|x|/2}}[B].$$

The result follows by taking $|x| \to \infty$.

Finally, the ergodicity of ϕ^0 can be proved similarly by taking A, B decreasing. $\square$

Lemma 3.7. *Fix $p \in [0,1]$ and $q \geq 1$. For $\phi^0_{p,q}$ or $\phi^1_{p,q}$, either there is no infinite cluster almost surely, or there exists a unique infinite cluster almost surely.*

Proof. The proof of Theorem 1.5 works and we only need to replace the independence between edges by the finite energy inequality (3.3). $\square$

The measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$ play specific roles in the theory. First, they are translation invariant and ergodic. They are extremal infinite-volume measures, in the sense that any infinite-volume measure ϕ for random-cluster model with $p \in [0,1]$ and $q \geq 1$ satisfies

$$\phi^0_{p,q}[A] \leq \phi_{p,q}[A] \leq \phi^1_{p,q}[A], \quad \forall \text{ increasing event } A. \tag{3.5}$$

Thus, if $\phi^0_{p,q} = \phi^1_{p,q}$, then there is a unique infinite-volume measure.

3.3 Phase Transition

Theorem 3.8. *Fix $q \geq 1$. There exists a critical point $p_c = p_c(q) \in [0,1]$ such that*

- *For $p > p_c$, any infinite-volume measure has an infinite cluster almost surely.*
- *For $p < p_c$, any infinite-volume measure has no infinite cluster almost surely.*

Lemma 3.9. *Fix $p \in [0,1]$ and $q \geq 1$. We have $\phi^0_{p,q} = \phi^1_{p,q}$ for all but countably many values of p.*

Proof. We first explain that the following equation implies $\phi^0 = \phi^1$:

$$\phi^0[\omega(e) = 1] = \phi^1[\omega(e) = 1]. \tag{3.6}$$

Assume (3.6) holds. Since $\phi^0_{\Lambda_n}$ is stochastically dominated by $\phi^1_{\Lambda_n}$, there exists a coupling $\mathbb{P}_n$ of the two measures such that $\mathbb{P}_n[\omega_0 \le \omega_1] = 1$ and the marginal of ω_0 is $\phi^0_{\Lambda_n}$ and the marginal of ω_1 is $\phi^1_{\Lambda_n}$. Suppose A is an increasing event depending only on edges inside Λ_N. We have, for $n \ge N$,

$$\begin{aligned} 0 \le \phi^1_{\Lambda_n}[A] - \phi^0_{\Lambda_n}[A] &= \mathbb{P}_n[\omega_1 \in A, \omega_0 \notin A] \\ &\le \mathbb{P}_n[\exists e \in \Lambda_N : \omega_1(e) = 1, \omega_0(e) = 0] \\ &\le \sum_{e \in \Lambda_N} \mathbb{P}_n[\omega_1(e) = 1, \omega_0(e) = 0] \\ &= \sum_{e \in \Lambda_N} \left(\phi^1_{\Lambda_n}[\omega(e) = 1] - \phi^0_{\Lambda_n}[\omega(e) = 0]\right). \end{aligned}$$

Let $n \to \infty$, we have $\phi^1[A] = \phi^0[A]$. Thus, (3.6) implies $\phi^0[A] = \phi^1[A]$ for any increasing event A depending on finitely many edges. This gives $\phi^0 = \phi^1$.

To show the conclusion, we only need to argue that (3.6) holds for all but countably many values of p. The proof needs the following conclusion the proof of which can be found in [FV18, Chapter 4 Theorem 4.6]: there exists a quantity, called the free energy, such that

$$f(p, q) = \lim_n \frac{1}{\#E(\Lambda_n)} \log \mathcal{Z}^{\xi}_{p,q,\Lambda_n}, \tag{3.7}$$

where the convergence holds uniformly in the choice of the boundary conditions ξ.

We will use this result to show the conclusion. Define the variable $\pi(p) = \log(p/(1-p))$ and also set $p_\pi = e^\pi/(1+e^\pi)$. Set

$$\begin{aligned} \tilde{f}^{\xi}_n(\pi, q) &= \frac{1}{\#E(\Lambda_n)} \log \mathcal{Z}^{\xi}_{p_\pi,q,\Lambda_n} - \log(1+e^\pi) = \frac{1}{\#E(\Lambda_n)} \log \sum_{\omega} e^{\pi o(\omega)} q^{k^{\xi}(\omega)}, \\ \tilde{f}(\pi, q) &= f(p_\pi, q) - \log(1+e^\pi). \end{aligned}$$

Differentiate $\tilde{f}^{\xi}_n(\pi, q)$ in π, we find

$$\partial_\pi \tilde{f}^{\xi}_n(\pi, q) = \frac{1}{\#E(\Lambda_n)} \sum_{e \in \Lambda_n} \phi^{\xi}_{p_\pi,q,\Lambda_n}[\omega(e) = 1].$$

The right hand-side is increasing in p and hence increasing in π by FKG. This implies that $\tilde{f}^{\xi}_n(\pi, q)$ is convex in π. Passing to the limit, the function $\tilde{f}(\pi, q)$ is also convex in π. This gives that $\tilde{f}(\pi, q)$ is differentiable in π for all but countably many points.

Let us prove that $\phi^0_{p_\pi,q}[\omega(e) = 1] = \phi^1_{p_\pi,q}[\omega(e) = 1]$ at values of π for which $\tilde{f}(\pi, q)$ is differentiable. Recall a standard result for convex functions: If (g_n) is a sequence of convex functions converging pointwise to a function g which is differentiable at x, then $\lim_n \partial^+ g_n(x) = \lim_n \partial^- g_n(x) = g'(x)$. Applying this result to the sequence $\tilde{f}^1_n(\pi, q)$ and $\tilde{f}^0_n(\pi, q)$ respectively, we deduce that

$$\begin{aligned} \partial_\pi \tilde{f}(\pi, q) &= \lim_n \frac{1}{\#E(\Lambda_n)} \sum_{e \in \Lambda_n} \phi^1_{p_\pi,q,\Lambda_n}[\omega(e) = 1] = \phi^1_{p_\pi,q}[\omega(e) = 1]; \\ \partial_\pi \tilde{f}(\pi, q) &= \lim_n \frac{1}{\#E(\Lambda_n)} \sum_{e \in \Lambda_n} \phi^0_{p_\pi,q,\Lambda_n}[\omega(e) = 1] = \phi^0_{p_\pi,q}[\omega(e) = 1]. \end{aligned}$$

We obtain (3.6) at any point of differentiability in π of $\tilde{f}(\pi, q)$. $\square$

Proof of Theorem 3.8. Define

$$p_c := \inf\{p \in [0,1] : \phi^0_{p,q}[0 \longleftrightarrow \infty] > 0\}.$$

Since the event $\{0 \longleftrightarrow \infty\}$ is increasing, we deduce $\phi^0_{p,q}[0 \longleftrightarrow \infty] > 0$ for $p > p_c$. Ergodicity in Lemma 3.6 implies that, for $p > p_c$,

$$\phi^0_{p,q}[\exists \text{ infinite cluster}] = 1.$$

Combing with (3.5), we obtain the conclusion for any infinite-volume measure with $p > p_c$.

For $p < p_c$, Lemma 3.9 implies that there exists $\tilde{p} \in (p, p_c)$ such that there exists a unique infinite-volume measure at $\tilde{p}$. Then for any infinite-volume measure $\phi_{p,q}$,

$$\phi_{p,q}[\exists \text{ infinite cluster}] \le \phi^1_{p,q}[\exists \text{ infinite cluster}] \le \phi^1_{\tilde{p},q}[\exists \text{ infinite cluster}] = \phi^0_{\tilde{p},q}[\exists \text{ infinite cluster}] = 0.$$

This gives the conclusion for $p < p_c$. □

3.4 Computation of the critical value on $\mathbb{Z}^2$

Theorem 3.10. *Consider the random-cluster model on $\mathbb{Z}^2$ with cluster-weight $q \ge 1$. The critical value p_c is given by*

$$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

Recall that we have introduced the dual graph and the dual configuration in Section 1.2. For a configuration $\omega \in \{0,1\}^{E(G)}$, the dual configuration $\omega^* \in \{0,1\}^{E(G^*)}$ is defined by

$$\omega^*(e^*) = 1 - \omega(e), \quad \forall e \in E(G).$$

Set $p^* = p^*(p,q)$ satisfying

$$\frac{pp^*}{(1-p)(1-p^*)} = q.$$

Proposition 3.11. *Let $\xi \in \{0,1\}^{E(\mathbb{Z}^2)\setminus E(G)}$. The dual configuration of the random-cluster model on G with parameters (p,q) and boundary conditions ξ is the random-cluster model with parameters (p^*,q) on G^* with boundary conditions ξ^* where $\xi^*(e^*) = 1 - \xi(e)$ for $e \notin E(G)$.*

Proof. Suppose ω^* is the dual configuration of ω, then the weight of ω^* is given by

$$\mathbb{P}[\omega^*] = \phi^\xi_{p,q,G}[\omega] \propto \left(\frac{p}{1-p}\right)^{o(\omega)} q^{k^\xi(\omega)}.$$

It is clear that $o(\omega) = c(\omega^*) = \#E(G) - o(\omega^*)$. To derive the relation between $k^\xi(\omega)$ and ω^*, we need Euler's formula. Let $f(\omega^\xi)$ be the number of faces in ω^ξ. Using Euler's formula, we find

$$o(\omega) + k(\omega^\xi) + 1 = \#V(G) + f(\omega^\xi).$$

Note that, the connected components of $(\omega^*)^{\xi^*}$ correspond exactly to faces of ω^ξ. Thus

$$k(\omega^\xi) = C_\xi + k^{\xi^*}(\omega^*) + o(\omega^*),$$

where C_ξ is a constant not depending on ω. These imply that

$$\mathbb{P}[\omega^*] \propto \left(\frac{p}{1-p}\right)^{o(\omega)} q^{k^\xi(\omega)} \propto \left(\frac{p}{1-p}\right)^{-o(\omega^*)} q^{k^{\xi^*}(\omega^*)+o(\omega^*)} = \left(\frac{(1-p)q}{p}\right)^{o(\omega^*)} q^{k^{\xi^*}(\omega^*)}.$$

This gives the conclusion. □

Fix $q \geq 1$ and introduce

$$p_{sd}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

This is the unique value of p satisfying $p^*(p,q) = p$.

Lemma 3.12. *Fix $q \geq 1$, we have*

$$\phi^0_{p_{sd}(q),q}[0 \leftrightarrow \infty] = 0.$$

Proof. If $\phi^0_{p_{sd}(q),q}[0 \leftrightarrow \infty] > 0$, we have infinite cluster under $\phi^0_{p_{sd},q}$. Then the dual model $\phi^1_{p_{sd},q}$ also have infinite cluster. We can repeat the same proof of Lemma 1.10 to argue that there are at least two infinite clusters under $\phi^0_{p_{sd},q}$ with positive chance where we only need to replace the edge independence by the finite energy property (3.3). This contradicts Lemma 3.7. $\square$

Proof of Theorem 3.10. Lemma 3.12 implies that $p_c \geq p_{sd}$. Assume $p_c > p_{sd}$. We can repeat the proof of Theorem 1.9 as long as we have the exponential decay for subcritical case. Indeed, the exponential decay holds, see Theorem 3.13. $\square$

Theorem 3.13. *Consider the random-cluster model on $\mathbb{Z}^2$ with cluster-weight $q \geq 1$.*

- *If $p < p_c$, then there exists $c = c(p) > 0$ such that for every $n \geq 1$, $\phi^1_{p,q,\Lambda_n}[0 \longleftrightarrow \partial\Lambda_n] \leq e^{-cn}$.*
- *If $p > p_c$, then there exists $C > 0$ such that $\phi^1_{p,q}[0 \longleftrightarrow \infty] \geq C(p - p_c)$.*

Proof. [DCRT19, Theorem 1.2]. $\square$

3.5 Continuous/Discontinuous phase transition

To end this section, we discuss the continuity of the phase transition. It turns out that the continuity depends on the cluster-weight q.

Theorem 3.14. *Fix $1 \leq q \leq 4$, we have $\phi^1_{p_c,q}[0 \longleftrightarrow \infty] = 0$.*

Proof. [DCST17, Theorem 3]. $\square$

Theorem 3.15. *Fix $q > 4$, we have $\phi^1_{p_c,q}[0 \longleftrightarrow \infty] > 0$, and $\phi^0_{p_c,q}[0 \longleftrightarrow \infty] = 0$.*

Proof. [DCGH$^+$16, Theorem 1.2] and [**?**, Theorem 1]. $\square$

For the random-cluster model in $\mathbb{Z}^2$, we say that we have continuous phase transition with $1 \leq q \leq 4$ and discontinuous phase transition with $q > 4$. As a consequence of Theorem 3.14, we have the uniqueness of infinite-volume measure.

Theorem 3.16. *Fix $1 \leq q \leq 4$, we have $\phi^1_{p,q} = \phi^0_{p,q}$ and hence this is the unique infinite-volume measure.*

Proof. Combining Lemma 3.17 and Corollary 3.18. $\square$

Lemma 3.17. *Fix $q \geq 1$. If $\phi^1_{p,q}[0 \leftrightarrow \infty] = 0$, we have $\phi^1_{p,q} = \phi^0_{p,q}$.*

Proof. We fix p, qin the proof and hence eliminate them from the notations. To show $\phi^1 = \phi^0$, it is sufficient to show (3.6) for an edge e that is adjacent to the origin. For $n \leq m$, consider the event $\{0 \not\leftrightarrow \partial\Lambda_n\}$ under the measure $\phi^1_{\Lambda_m}$. Denote by $\mathcal{C}$ the cluster inside Λ_n connecting to $\partial\Lambda_n$. Then the event $\{0 \not\leftrightarrow \partial\Lambda_n\}$ is the same as $\{0 \notin \mathcal{C}\}$. For any realization A of $\mathcal{C}$ such that $0 \notin A$, define U to be the connected component of Λ_n containing the origin obtained by removing from Λ_n all the edges in A and all the edges adjacent to A. Then we have

$$\begin{aligned}
\phi^1_{\Lambda_m}[\omega(e)=1, 0 \not\leftrightarrow \partial\Lambda_n] &= \sum_{A: 0\notin A} \phi^1_{\Lambda_m}[\omega(e)=1, \mathcal{C}=A] \\
&= \sum_{A: 0\notin A} \phi^1_{\Lambda_m}[\mathcal{C}=A]\phi^1_{\Lambda_m}[\omega(e)=1 \,|\, \mathcal{C}=A] \\
&= \sum_{A: 0\notin A} \phi^1_{\Lambda_m}[\mathcal{C}=A]\phi^0_U[\omega(e)=1] \\
&\le \sum_{A: 0\notin A} \phi^1_{\Lambda_m}[\mathcal{C}=A]\phi^0[\omega(e)=1] = \phi^1_{\Lambda_m}[0 \not\leftrightarrow \partial\Lambda_n]\phi^0[\omega(e)=1].
\end{aligned}$$

As $m \to \infty$, we have

$$\phi^1[\omega(e)=1, 0 \not\leftrightarrow \partial\Lambda_n] \le \phi^0[\omega(e)=1].$$

As $n \to \infty$, since $\phi^1[0 \not\leftrightarrow \infty] = 1$, we have

$$\phi^1[\omega(e)=1] \le \phi^0[\omega(e)=1],$$

which implies (3.6) and completes the proof. □

Corollary 3.18. *Fix $q \ge 1$. The unique edge-weight $p \in [0,1]$ for which there can exist distinct infinite-volume measures is $p_{sd}(q)$.*

Proof. We fix $q \ge 1$ and hence eliminate it from the notation. Lemma 3.12 gives $\phi^0_{p_{sd}}[0 \leftrightarrow \infty] = 0$. For any $p < p_{sd}$, by Lemma 3.9, there exists $\tilde{p} \in (p, p_{sd})$ such that $\phi^1_{\tilde{p}} = \phi^0_{\tilde{p}}$. Then

$$\phi^1_p[0 \leftrightarrow \infty] \le \phi^1_{\tilde{p}}[0 \leftrightarrow \infty] = \phi^0_{\tilde{p}}[0 \leftrightarrow \infty] \le \phi^0_{p_{sd}}[0 \leftrightarrow \infty] = 0.$$

Combining with Lemma 3.17, we have the uniqueness of the infinite-volume measure at (p,q).

For $p > p_{sd}$, we have $p^* < p_{sd}$. The dual measures of $\phi^0_{p,q}$ and $\phi^1_{p,q}$ are $\phi^1_{p^*,q}$ and $\phi^0_{p^*,q}$ respectively. Since $p^* < p_{sd}$, we see $\phi^1_{p^*,q} = \phi^0_{p^*,q}$ and hence $\phi^0_{p,q} = \phi^1_{p,q}$. □

3.6 Exercises

Exercise 3.1. *Proposition 3.5 and Lemmas 3.6 and 3.7 hold for the random-cluster model in $\mathbb{Z}^d$ with $d \ge 2$.*

Exercise 3.2 (Differential formula)**.** *Let A be an increasing event depending on edges in G only. Let $I_A(e)$ be the influence of the edge e defined by*

$$I_A(e) := \phi^\xi_{p,q,G}[A \,|\, \omega(e)=1] - \phi^\xi_{p,q,G}[A \,|\, \omega(e)=0].$$

- *Show that*

$$I_A(e) \ge \phi^\xi_{p,q,G}[e \text{ is pivotal for } A].$$

- *Show that*

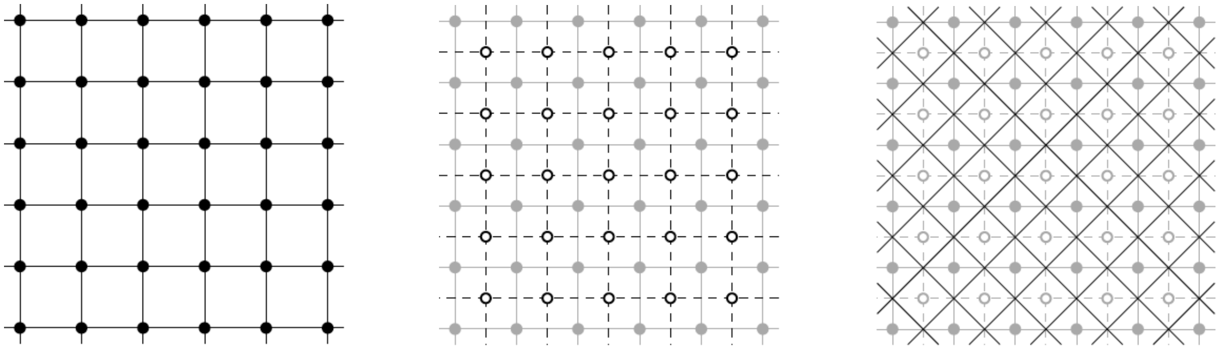
$$\frac{d}{dp}\phi^\xi_{p,q,G}[A] = \sum_e I_A(e).$$

SRT 5. *Random-cluster model.*

Please give a report on Theorem 3.13, Theorem 3.14, Theorem 3.15.

4 Conformal Invariance for FK-Ising Model

In this section, we focus on FK-Ising model, i.e. random cluster model with $q = 2$.



(a) The square lattice. (b) The dual lattice. (c) The medial lattice.

Figure 4.1: The lattices.

The *dual square lattice* $(\mathbb{Z}^2)^*$ is the dual graph of $\mathbb{Z}^2$. The vertex set is $(1/2, 1/2) + \mathbb{Z}^2$ and the edges are given by nearest neighbors. The vertices and edges of $(\mathbb{Z}^2)^*$ are called dual-vertices and dual-edges. In particular, for each edge e of $\mathbb{Z}^2$, it is associated to a dual edge, denoted by e^*. The dual edge e^* crosses e in the middle. The *medial lattice* $(\mathbb{Z}^2)^\diamond$ is the graph with the centers of edges of $\mathbb{Z}^2$ as vertex set, and edges connecting nearest vertices. This lattice is a rotated and rescaled version of $\mathbb{Z}^2$, see Fig 4.1. The vertices and edges of $(\mathbb{Z}^2)^\diamond$ are called medial-vertices and medial-edges. We identify the faces of $(\mathbb{Z}^2)^\diamond$ with the vertices of $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^*$. A face of $(\mathbb{Z}^2)^\diamond$ is said to be black if it corresponds to a vertex of $\mathbb{Z}^2$ and white if it corresponds to a vertex of $(\mathbb{Z}^2)^*$. We denote the square lattice by $\mathbb{L}$, and denote the dual lattice and the medial lattice by $\mathbb{L}^*$ and $\mathbb{L}^\diamond$ respectively.

For $\delta > 0$, the square lattice $\sqrt{2}\delta\mathbb{L}$ of mesh-size $\sqrt{2}\delta$ will be denoted by $\mathbb{L}_\delta$. The dual and medial lattices extend to this context and they are denoted by $\mathbb{L}_\delta^*$ and $\mathbb{L}_\delta^\diamond$. Note that the medial lattice $\mathbb{L}_\delta^\diamond$ has mesh-size δ. For a simply connected domain Ω, we set $\Omega_\delta = \Omega \cap \mathbb{L}_\delta$. The edges connecting sites of Ω_δ are those included in Ω.

Let $(\Omega; a, b)$ be a Dobrushin domain. Let $\Omega_\delta^\diamond$ be the medial graph of Ω_δ composed of all the vertices of $\mathbb{L}_\delta^\diamond$ bordering a black face associated to Ω_δ.[3] Let a_δ, b_δ be two vertices of $\partial\Omega_\delta^\diamond$ close to a and b and require that b_δ is the southeast corner of a black face.

4.1 Fermionic observable

Fix a Dobrushin domain $(\Omega; a, b)$ and consider a configuration ω together with its dual-configuration ω^*. The *Dobrushin boundary condition* is given by taking edges of (ba) to be open and the dual-edges of (a^*b^*) to be dual-open; in this case, we also say that the boundary condition along (ba) is wired and the boundary condition along (ab) is free. Through every vertex of $\Omega^\diamond$, there passes either an open edge of Ω or a dual open edge of Ω^*. Draw self-avoiding loops on $\Omega^\diamond$ as follows: a loop arriving at a vertex of the medial lattice always makes a $\pm\pi/2$ turn so as not to cross the open or dual open edges through this vertex. The loop representation contains loops together with a self-avoiding path going from $a^\diamond$ to $b^\diamond$, see Fig. 4.2. This curve is called the *exploration path* in $\Omega^\diamond$ from $a^\diamond$ to $b^\diamond$.

Lemma 4.1. *Consider FK-Ising model in $(\Omega_\delta; a_\delta, b_\delta)$ with Dobrushin boundary conditions. We have*

$$\phi_{(\Omega_\delta; a_\delta, b_\delta)}(\omega) = \frac{1}{Z} x^{o(\omega)} \sqrt{2}^{\ell(\omega)}, \quad \textit{where } x = \frac{p}{\sqrt{2}(1-p)},$$

[3]This definition is non-standard since we include medial vertices not associated to edges of Ω_δ.

and $\ell(\omega)$ is the number of loops in the loop configuration associated to ω.

Proof. Note that

$$\ell(\omega) = k(\omega) + k(\omega^*) - 1 = 2k(\omega) + o(\omega) + const.$$

$\square$

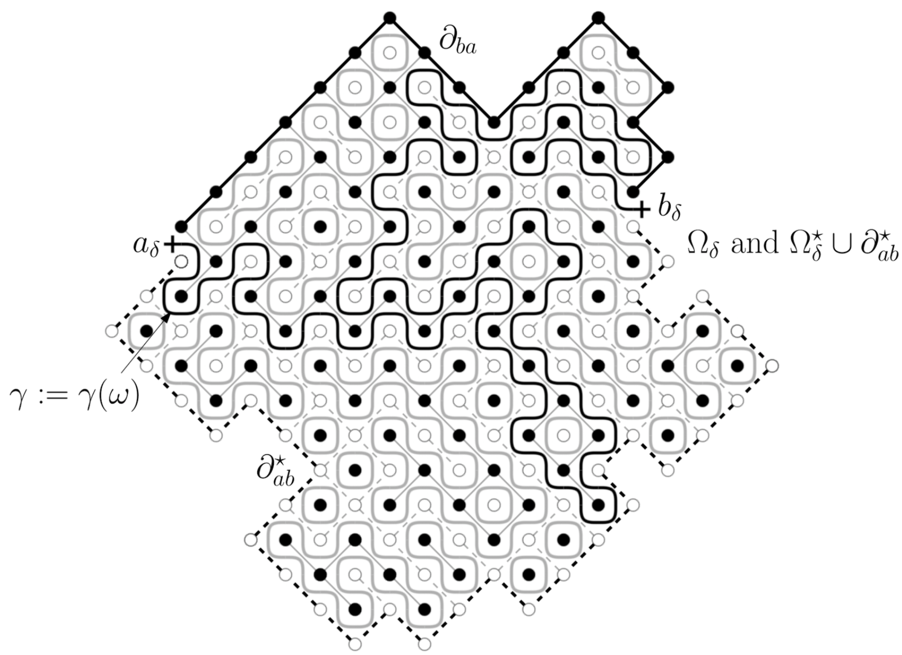


Figure 4.2

Definition 4.2. *The edge FK fermionic observable is defined on edges of $\Omega_\delta^\diamond$ by*

$$F_\delta(e) = \mathbb{E}\left[\mathbb{1}_{\{e\in\gamma\}} \exp\left(\frac{i}{2} W_\gamma(e, b_\delta)\right)\right],$$

where $W_\gamma(e, b_\delta)$ denotes the winding between the center of e and $b_\delta^\diamond$. The vertex FK fermionic observable is defined on vertices of $\Omega_\delta^\diamond \setminus \partial\Omega_\delta^\diamond$ by

$$F_\delta(v) = \frac{1}{2}\sum_{e\sim v} F_\delta(e),$$

where the sum is over the four medial edges having v as an endpoint.

The goal of this section is the following.

Theorem 4.3. *Fix a Dobrushin domain $(\Omega; a, b)$ and let $(\Omega_\delta; a_\delta, b_\delta)$ be a sequence of discrete Dobrushin domains converging to $(\Omega; a, b)$ in the Carathéodory sense. Consider the critical FK-Ising model on $(\Omega_\delta; a_\delta, b_\delta)$ with Dobrushin boundary conditions. Let F_δ be the vertex fermionic observable in $(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)$. Then, we have*

$$\frac{1}{\sqrt{2\delta}} F_\delta \to \sqrt{\phi'}, \quad \text{as } \delta \to 0, \quad \text{locally uniformly,}$$

where ϕ is any conformal map from Ω on to the strip $\mathbb{R} \times (0,1)$ sending a to $-\infty$ and b to $+\infty$.

4.2 Discrete complex analysis on graphs

In this section, we shall discuss how to discretize harmonic and holomorphic functions, and what are the properties of these discretizations. Since we need to consider scaling limits, we want to deal with discrete structure which converge to the continuous complex analysis as finer and finer graphs are taken.

Preharmonic functions

We first introduce the discretization of the Laplacian operator in the case of the square lattice $\mathbb{L}_\delta$. For $x \in \mathbb{L}_\delta$ and $h : \mathbb{L}_\delta \to \mathbb{R}$, define

$$\Delta_\delta h(x) = \frac{1}{4} \sum_{y:\ y \sim x} (h(y) - h(x)).$$

A function $h : \Omega_\delta \to \mathbb{R}$ is preharmonic (resp. pre-superharmonic, pre-subharmonic) if $\Delta_\delta h(x) = 0$ (resp. $\Delta_\delta h(x) \le 0$, $\Delta_\delta h(x) \ge 0$) for every $x \in \Omega_\delta$.

The classical relation between preharmonic function and simple random walk is the following: Let $(X_n)_{n\ge1}$ be a simple random walk on $\mathbb{L}_\delta$ killed at the first time it exits Ω_δ, then h is preharmonic if and only if $(h(X_n))_{n\ge1}$ is a martingale. Consequently, we have solution to the Dirichlet boundary value problem in the discrete.

Lemma 4.4. *Let Ω be simply connected. Given $f_\delta : \partial\Omega_\delta \to \mathbb{R}$, there exists a unique preharmonic function on Ω_δ such that $h_\delta = f_\delta$ on $\partial\Omega_\delta$.*

Proof. We first show the existence. Let $(X_n)_{n\ge1}$ be a simple random walk starting from $x \in \Omega_\delta$ and let τ be the first time that it hits $\partial\Omega_\delta$. Define

$$h_\delta(x) = \mathbb{E}[f_\delta(X_\tau)].$$

Then h_δ is harmonic on Ω_δ and $h_\delta = f_\delta$ on $\partial\Omega_\delta$.

Next, we show that preharmonic functions attain the maximum on the boundary. Let h be a preharmonic function on Ω_δ. Suppose it attains its maximum at $x_0 \in \Omega_\delta$, we will show that h is constant. As $\sum_{x\sim x_0} h(x) = 4h(x_0)$, we see that $h(x) = h(x_0)$ for all $x \sim x_0$. For any $y \in \Omega_\delta$, there exists a path $x_0 \sim x_1 \sim \cdots \sim x_n = y$ such that $x_i \in \Omega_\delta$. Repeating the previous argument, we find $h(y) = h(x_n) = \cdots = h(x_0)$. Therefore, h is constant.

Finally, let us show the uniqueness. Suppose there are two functions h_1 and h_2 such that they are preharmonic on Ω_δ and $h_1 = h_2 = f_\delta$ on $\partial\Omega_\delta$. Then $h_1 - h_2$ is preharmonic on Ω_δ, and thus it attains its maximum and its minimum on $\partial\Omega_\delta$. As $h_1 - h_2 = 0$ on $\partial\Omega_\delta$, we see that $h_1 - h_2 = 0$ on Ω_δ. $\square$

We have the convergence of the discrete Dirichlet problem solution to the continuous one.

Theorem 4.5. *Let $(\Omega; a, b)$ be a Dobrushin domain, f be a bounded continuous function on $\partial\Omega \setminus \{a, b\}$, and h be the unique harmonic function on Ω, continuous on $\overline{\Omega} \setminus \{a, b\}$, satisfying $h = f$ on $\partial\Omega \setminus \{a, b\}$. Let $(\Omega_\delta; a_\delta, b_\delta)$ be a sequence of discrete Dobrushin domains converging to $(\Omega; a, b)$ in the Carathéodory sense. Let $f_\delta : \partial\Omega_\delta \to \mathbb{R}$ be a sequence of uniformly bounded functions converging to f uniformly away from a and b. Let h_δ be the unique preharmonic function on Ω_δ such that $h_\delta = f_\delta$ on $\partial\Omega_\delta$. Then $h_\delta \to h$ locally uniformly as $\delta \to 0$.*

Proof. See e.g. [DC13, Theorem 8.9]. $\square$

Preholomorphic functions

Isaacs's definition of preholomorphic function involves the following discretization of the operator $\bar\partial = \partial_x + i\partial_y$. For a function $f : \mathbb{L}_\delta \to \mathbb{C}$, and $x \in \mathbb{L}_\delta^*$, define

$$\bar\partial_\delta f(x) = \frac{1}{2}(f(E) - f(W)) + \frac{i}{2}(f(N) - f(S)),$$

where N, E, S, W are the four vertices of $\mathbb{L}_\delta$ adjacent to the dual vertex x indexed in the obvious way. A function $f : \Omega_\delta \to \mathbb{C}$ is preholomorphic if $\bar\partial_\delta f(x) = 0$ for all $x \in \Omega_\delta^*$. The equation $\bar\partial_\delta f(x) = 0$ is called the Cauchy-Riemann equation at x.

Preholomorphic functions satisfy the following properties.

Lemma 4.6. *(1) Sums of preholomorphic functions are preholomorphic.*

(2) Discrete contour integrals vanish in simply connected domain.

(3) The primitive in simply connected domain is well-defined.

(4) If a family (f_δ) of preholomorphic functions on Ω_δ converges locally uniformly to f on Ω, then f is holomorphic.

Proof of Item (2). It suffices to show that the integral around a face vanishes. Suppose the face has center $x \in \mathbb{L}_\delta^*$ and denote the four vertices by N, E, S, W indexed in the obvious way. The integral of f around the face is the following:

$$\begin{aligned}&\frac{1}{2}(f(S)+f(E))\delta(1+i)+\frac{1}{2}(f(E)+f(N))\delta(-1+i)\\&+\frac{1}{2}(f(N)+f(W))\delta(-1-i)+\frac{1}{2}(f(S)+f(W))\delta(1-i)\\&=\delta 2i\bar\partial_\delta f(x)=0.\end{aligned}$$

□

Proof of Item (4). Morera's theorem. □

s-holomorphic functions

As explained in the previous section, the theory of preholomorphic functions starts like the continuum theory. Unfortunately, problems arrive quickly. For instance, the product of two preholomorphic functions is no longer preholomorphic in general. In order to partially overcome this difficulty, we introduce s-holomorphic functions (for spin-holomorphic), a notion that will be central in the study of the spin fermionic observables.

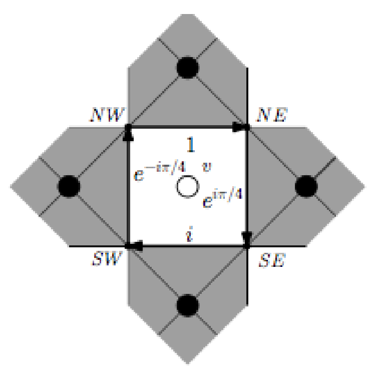


Figure 4.3: Lines $\ell(e)$ for medial edges around a white face.

To each edge e on the medial lattice, associate a direction $\ell(e)$, see Figure 4.3 for the definition. In other words, $\ell(e)$ has the same direction as $\sqrt{\bar e}$ when one view the oriented edge e as a complex number. A function f is s-holomorphic if for any edge e of $\Omega_\delta^\diamond$, we have

$$P_{\ell(e)}[f(x)] = P_{\ell(e)}[f(y)],$$

where x, y are the endpoints of e and P_ℓ is the orthogonal projection on the direction ℓ. The definition of s-holomorphic is not rotationally invariant: f is s-holomorphic if and only if $e^{i\pi/4}f(i\cdot)$ is s-holomorphic.

Proposition 4.7. *Any s-holomorphic function $f : \Omega_\delta^\diamond \to \mathbb{C}$ is preholomorphic on $\Omega_\delta^\diamond$.*

Proof. Let $f : \Omega_\delta^\diamond \to \mathbb{C}$ be a s-holomorphic function and let v be a vertex of $\mathbb{L}_\delta \cup \mathbb{L}_\delta^*$ (this is the vertex set of the dual graph of the medial lattice). Assume that $v \in \Omega_\delta^*$, the other case is similar. We aim to show

$$\bar{\partial}_\delta f(v) = \frac{1}{2}(f(SE) - f(NW)) + \frac{i}{2}(f(NE) - f(SW)) = 0.$$

The relation provided by the s-holomorphicity around the face centered at v is the following:

$$\begin{aligned} f(NW) + \overline{f(NW)} &= f(NE) + \overline{f(NE)}; \\ f(NE) + i\overline{f(NE)} &= f(SE) + i\overline{f(SE)}; \\ f(SE) - \overline{f(SE)} &= f(SW) - \overline{f(SW)}; \\ f(SW) - i\overline{f(SW)} &= f(NW) - i\overline{f(NW)}. \end{aligned}$$

Multiplying the second identity by $-i$, the third one by -1, the fourth one by i, and then summing the four identities, we obtain

$$0 = 2(1-i)\bar{\partial}_\delta f(v).$$

This gives the conclusion. ☐

The interest in s-holomorphic functions comes from the fact that a relevant discretization of $\Im \int f^2$ can be defined for s-holomorphic function f.

Theorem 4.8. *Let Ω be a simply connected domain. Suppose $f : \Omega_\delta^\diamond \to \mathbb{C}$ is an s-holomorphic function and $b_0 \in \Omega_\delta$. Then, there exists a unique function $H : \Omega_\delta \cup \Omega_\delta^* \to \mathbb{C}$ such that*

$$H(b_0) = 1, \quad \text{and} \quad H(b) - H(w) = \delta|P_{\ell(e)}[f(x)]|^2 (= \delta|P_{\ell(e)}[f(y)]|^2),$$

for every edge $e = (x, y)$ on $\Omega_\delta^\diamond$ bordered by a black face $b \in \Omega_\delta$ and a white face $w \in \Omega_\delta^*$. For two neighboring sites $b_1, b_2 \in \Omega_\delta$, with v being the medial vertex at the center of (b_1, b_2), we have

$$H(b_1) - H(b_2) = \frac{1}{2}\Im(f(v)^2(b_1 - b_2)). \tag{4.1}$$

The same relation holds for sites of Ω_δ^*. This legitimizes the fact that H is a discrete analogue of $\frac{1}{2}\Im \int f^2$.

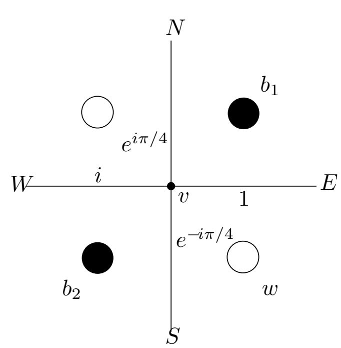


Figure 4.4

Proof of Theorem 4.8. One can construct the value of the function at a vertex by summing increments along an arbitrary path from b_0 to this vertex. The only thing to check is that the value obtained in this way does not depend on the chosen path. Since $\Omega_\delta^\diamond$ is simply connected, it is sufficient to check it for elementary square contours around each medial vertex v. Therefore, we only need to check

$$|P_{\ell(S)}[f(v)]|^2 - |P_{\ell(E)}[f(v)]|^2 = |P_{\ell(W)}[f(v)]|^2 - |P_{\ell(N)}[f(v)]|^2, \tag{4.2}$$

where N, E, S, W are the four medial edges around v indexed in the obvious way, see Figure 4.4. Note that $\ell(N), \ell(S)$ are orthogonal, and $\ell(E), \ell(W)$ are orthogonal. Thus

$$|P_{\ell(N)}[f(v)]|^2 + |P_{\ell(S)}[f(v)]|^2 = |f(v)|^2 = |P_{\ell(E)}[f(v)]|^2 + |P_{\ell(W)}[f(v)]|^2.$$

This gives (4.2).

It remains to show (4.1). We may assume $b_1 - b_2 = \delta(1+i)$ and let $w = b_2 + \delta$. Denote by S the edge between b_2 and w and by E the edge between b_1 and w. From definition, we have

$$\begin{aligned} H(b_1) - H(w) =& \delta|P_{\ell(E)}[f(v)]|^2 = \delta(\Re f(v))^2, \\ H(b_2) - H(w) =& \delta|P_{\ell(S)}[f(v)]|^2 = \frac{\delta}{2}(\Re f(v) - \Im f(v))^2. \end{aligned}$$

Taking the difference, we have

$$H(b_1) - H(b_2) = \frac{\delta}{2}((\Re f(v))^2 + 2(\Re f(v))(\Im f(v)) - (\Im f(v))^2) = \frac{1}{2}\Im(f(v)^2(b_1 - b_2)).$$

□

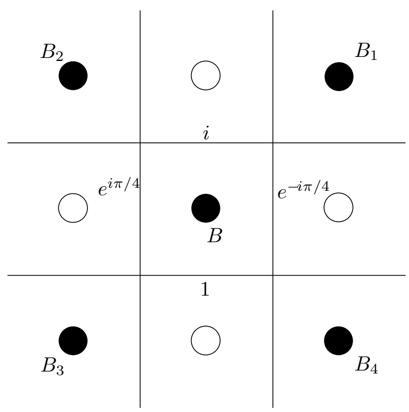


Figure 4.5

Proposition 4.9. *Denote by $H^\bullet$ the restriction of H to Ω_δ (black faces) and by H° the restriction of H to Ω_δ^* (white faces). If f is s-holomorphic, then $H^\bullet$ is subharmonic and H° is superharmonic.*

Proof. We only prove for $H^\bullet$. Suppose B is a vertex of $\Omega_\delta \backslash \partial\Omega_\delta$. We aim to show that the sum of increments of $H^\bullet$ between B and its four neighbors is positive, see Figure 4.5. We denote by B_1, B_2, B_3, B_4 the four neighbors in counterclockwise order such that $B_1 = B + \delta(1+i)$. Denote by v_k the center between B and B_k and denote $a_k = \Re f(v_k)$ and $b_k = \Im f(v_k)$ for $k = 1, 2, 3, 4$. From (4.1), we have

$$H(B_k) - H(B) = \frac{1}{2}\Im(f(v_k)^2(B_k - B)), \quad k = 1, 2, 3, 4. \tag{4.3}$$

As f is s-holomorphic, we have

$$a := a_3 = a_4, \quad b := b_1 = b_2, \quad c := \frac{a_2 + b_2}{\sqrt{2}} = \frac{a_3 + b_3}{\sqrt{2}}, \quad d := \frac{a_4 - b_4}{\sqrt{2}} = \frac{a_1 - b_1}{\sqrt{2}}.$$

Thus,

$$a_1 = b + d\sqrt{2}, \quad a_2 = c\sqrt{2} - b, \quad a_3 = a_4 = a, \quad b_1 = b_2 = b, \quad b_3 = c\sqrt{2} - a, \quad b_4 = a - d\sqrt{2}.$$

Plugging into (4.3), we have

$$\begin{aligned}\frac{2}{\delta}\sum_{k=1}^{4}(H(B_k) - H(B)) =& a_1^2 + 2a_1b_1 - b_1^2 + a_2^2 - 2a_2b_2 - b_2^2 - a_3^2 - 2a_3b_3 + b_3^2 - a_4^2 + 2a_4b_4 + b_4^2 \\ =& 4(a^2 + b^2 + c^2 + d^2) + 4\sqrt{2}(bd - bc - ac - ad) \geq 0.\end{aligned}$$

□

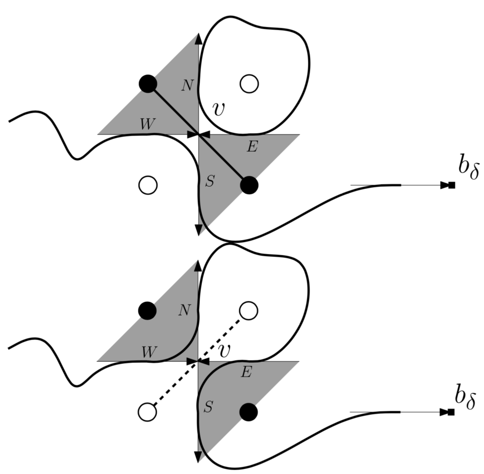
4.3 Proof of Theorem 4.3

In this section, we fix $p = \frac{\sqrt{2}}{1+\sqrt{2}}$. Then, $x = 1$ in Lemma 4.1.

Lemma 4.10. *Consider a medial vertex $v \in \Omega_\delta^\diamond \setminus \partial\Omega_\delta^\diamond$. We have*

$$F_\delta(N) - F_\delta(S) = i\left(F_\delta(E) - F_\delta(W)\right),$$

where N, E, S, W are the four adjacent edges indexed in the obvious way.



configuration	W	E	N	S
ω	W_ω	0	0	$e^{i\pi/4}W_\omega$
$s(\omega)$	$W_\omega/\sqrt{2}$	$e^{i\pi/2}W_\omega/\sqrt{2}$	$e^{-i\pi/4}W_\omega/\sqrt{2}$	$e^{i\pi/4}W_\omega/\sqrt{2}$

Figure 4.6

Proof. We may assume v as in Figure 4.6. Let e be a medial edge and set

$$e_\omega = \phi_{(\Omega_\delta; a_\delta, b_\delta)}[\omega]\,\mathbb{1}_{\{e\in\gamma\}}\exp\left(\frac{i}{2}W_\gamma(e, b_\delta)\right)$$

the contribution of the configuration ω to $F_\delta(e)$. Denote by s the involution on the space of configurations which switches the state of the primal edge corresponding to v. Then we have

$$F_\delta(e) = \sum_\omega e_\omega = \frac{1}{2}\sum_\omega (e_\omega + e_{s(\omega)}).$$

It suffices to show that, for all ω,

$$N_\omega + N_{s(\omega)} - S_\omega - S_{s(\omega)} = i\left(E_\omega + E_{s(\omega)} - W_\omega - W_{s(\omega)}\right). \tag{4.4}$$

There are three cases.

- Case 1. The exploration path $\gamma(\omega)$ does not go through any of the edges adjacent to v. Then neither does $s(\omega)$. Thus the terms in (4.4) all vanish and (4.4) holds.
- Case 2. The exploration path $\gamma(\omega)$ goes through two edges around v. We may assume that it enters v through W and leaves through S, and the other cases can be analyzed similarly. In such case, the primal edge corresponding to v is open. We compute the contribution of all the edges W, E, N, S of ω and $s(\omega)$ in terms of W_ω. See Figure 4.6. Then we may check that (4.4) holds.
- Case 3. The exploration path $\gamma(\omega)$ goes through the four medial edges around v. The computation is the same as in the second case.

□

Lemma 4.11. *The vertex fermionic observable F_δ is s-holomorphic.*

Proof. We may assume that the medial edge corresponding to b_δ is horizontal pointing to the right. In such case, it is clear that $F_\delta(e)$ belongs to $\ell(e)$. Consider a medial vertex v and the four edges N, E, S, W adjacent to it. As $F_\delta(e)$ belongs to $\ell(e)$, we may write

$$F_\delta(N) = (1-i)a_N, \quad F_\delta(E) = ia_E, \quad F_\delta(S) = (1+i)a_S, \quad F_\delta(W) = a_W, \quad \text{with } a_N, a_E, a_S, a_W \in \mathbb{R}.$$

From Lemma 4.10, we have $a_E = a_S - a_N$ and $a_W = a_N + a_S$. In particular,

$$F_\delta(v) = F_\delta(N) + F_\delta(S) = F_\delta(E) + F_\delta(W).$$

Note that $F_\delta(N)$ and $F_\delta(S)$ are orthogonal. Thus $F_\delta(N)$ is the projection of $F_\delta(v)$ to $\ell(N)$. This holds in general: $F_\delta(e)$ is the projection of $F_\delta(v)$ to $\ell(e)$ when v is an endpoint of e. □

Let A be the black face bordering a_δ. Since $\frac{1}{\sqrt{2\delta}}F_\delta$ is s-holomorphic, Theorem 4.8 defines a function $H_\delta : \Omega_\delta \cup \Omega_\delta^* \to \mathbb{R}$ such that

$$H(A) = 1, \quad \text{and} \quad H_\delta(B) - H_\delta(W) = |P_{\ell(e)}[F_\delta(x)]|^2 = |P_{\ell(e)}[F_\delta(y)]|^2,$$

for the medial edge $e = (x, y)$ bordered by a black face $B \in \Omega_\delta$ and a white face $W \in \Omega_\delta^*$. Recall that its restriction $H_\delta^\bullet$ to Ω_δ is subharmonic and its restriction H_δ° to Ω_δ^* is superharmonic.

Lemma 4.12.

- *The subharmonic function $H_\delta^\bullet$ is equal to 1 on (ba), and it converges to 0 on (ab) uniformly away from a and b.*
- *The superharmonic function H_δ° is equal to 0 on (a^*b^*), and it converges to 1 on (b^*a^*) uniformly away from a and b.*

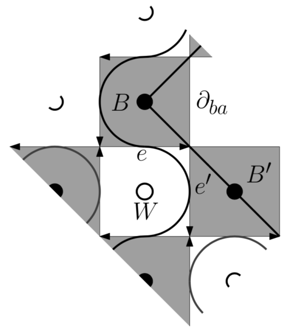


Figure 4.7

Proof. We first show that $H_\delta^\bullet$ is constant along (ba). The notations are explained in Figure 4.7. Then we have

$$H_\delta^\bullet(B) - H_\delta^\bullet(B') = |F_\delta(e)|^2 - |F_\delta(e')|^2 = 0.$$

Thus, $H_\delta^\bullet$ is constant along (ba). Since $H_\delta^\bullet(A) = 1$, it is one along (ba). Similarly, H_δ° is 0 along (a^*b^*).

Next, we consider $H_\delta^\bullet$ along (ab). We have

$$H_\delta^\bullet(B) = H_\delta^\circ(W) + |F_\delta(e)|^2 = |F_\delta(e)|^2 = \phi_{(\Omega_\delta;a_\delta,b_\delta)}[e \in \gamma]^2.$$

Suppose B is at distance r from (ba), and let $U_\delta = (B + [-r,r]^2) \cap \mathbb{L}_\delta$. Then we have

$$H_\delta^\bullet(B) = \phi_{(\Omega_\delta;a_\delta,b_\delta)}[e \in \gamma]^2 \leq \phi^1_{U_\delta}[B \leftrightarrow \partial U_\delta]^2 \to 0, \quad \text{as } \delta \to 0.$$

Similarly, we have the convergence of H_δ° along (b^*a^*). □

Proposition 4.13. *The sequence $(H_\delta)_{\delta>0}$ converges to $\Im\phi$ locally uniformly.*

Proof. Let $h_\delta^\bullet$ (resp. h_δ°) be the preharmonic function with the same boundary data as $H_\delta^\bullet$ (resp. H_δ°). Since $H_\delta^\bullet$ is subharmonic, we have $H_\delta^\bullet \leq h_\delta^\bullet$. Since H_δ° is superharmonic, we have $H_\delta^\circ \geq h_\delta^\circ$. Suppose $K \subset \Omega$ is compact. Let $b_\delta \in K_\delta$ and $w_\delta \in K_\delta^*$ be neighbors. We have

$$h_\delta^\circ(w_\delta) \leq H_\delta^\circ(w_\delta) \leq H_\delta^\bullet(b_\delta) \leq h_\delta^\bullet(b_\delta).$$

Applying Theorem 4.5 to h_δ° and to $h_\delta^\bullet$, we obtain the conclusion. □

Proof of Theorem 4.3. We assume the tightness of $(\frac{1}{\sqrt{2\delta}}F_\delta)_{\delta>0}$ is known. Suppose $(\frac{1}{\sqrt{2\delta_n}}F_{\delta_n})$ is a convergent subsequence and denote the limit by f. Since F_{δ_n} is preholomorphic, the limit f is holomorphic.

For $x, y \in \Omega$, we have

$$H_{\delta_n}(y) - H_{\delta_n}(x) = \frac{1}{2\delta_n}\Im\int_x^y F_{\delta_n}(z)^2 dz.$$

Since $F_{\delta_n} \to f$ locally uniformly, the RHS converges to $\Im\int_x^y f(z)^2 dz$. From Proposition 4.13, the LHS converges to $\Im(\phi(y) - \phi(x))$. Both $\int_x^y f(z)^2 dz$ and $\phi(y) - \phi(x)$ are holomorphic functions in y. Thus

$$\phi(y) - \phi(x) = const + \int_x^y f(z)^2 dz.$$

Therefore, $f = \sqrt{\phi'}$. □

4.4 Exercises

Exercise 4.1. *Show that the restriction of a continuous holomorphic function to* $\mathbb{L}_\delta$ *satisfies the discrete Cauchy-Riemann equations up to* $O(\delta^3)$.

Exercise 4.2. *Show that*

$$\frac{1}{\sqrt{2}\delta}\bar{\partial}_\delta, \quad \frac{1}{2\delta^2}\Delta_\delta$$

converge (as $\delta \to 0$*) to* $\bar{\partial}, \Delta$ *in the sense of distributions.*

SRT 6. *Convergence of interfaces in FK-Ising model.*

Please give a report on FK-Ising model discussing the following topics.

(1) convergence of the interface to SLE(16/3) in Dobrushin domain.

(2) convergence of the interface in polygon with alternating boundary conditions.

References: [CS12, CDCH$^+$14, Izy20].

5 Ising Model

The (spin) Ising model can be defined on any graph, and we will focus on the square lattice. Let G be a finite subgraph of $\mathbb{Z}^d$ and fix some boundary conditions $b \in \{\ominus, \oplus\}^{\partial G}$. The probability measure $\mu^b_{\beta,G}$ of the Ising model on G with inverse-temperature $\beta > 0$ and boundary conditions b is defined by

$$\mu^b_{\beta,G}[\sigma] = \frac{1}{\mathcal{Z}^b_{\beta,G}} \exp\left(\beta \sum_{x\sim y} \sigma_x \sigma_y\right),$$

for every configuration $\sigma \in \{\ominus, \oplus\}^G$ such that $\sigma = b$ on ∂G. The constant $\mathcal{Z}^b_{\beta,G}$ is a normalizing constant, referred to as the partition function. We list a few boundary conditions that will come back later in the notes.

- $\oplus$-boundary conditions. This is the case when $b_x = \oplus$ for all $x \in \partial G$. The corresponding Ising model on G is denoted by $\mu^\oplus_{\beta,G}$. Similarly, the Ising model with $\ominus$-boundary conditions is denoted by $\mu^\ominus_{\beta,G}$.
- free-boundary conditions. One can define the Ising model without boundary conditions, also called free boundary conditions. The corresponding Ising model on G is denoted by $\mu^f_{\beta,G}$.
- Dobrushin-boundary conditions. Let $(\Omega; a, b)$ be a Dobrushin domain. Define $\partial_e\Omega = \{x \in \mathbb{Z}^2 \backslash V(\Omega) : \exists y \in V(\Omega) \text{ such that } (x,y) \in E(\mathbb{Z}^2)\}$. Assume that $\partial_e\Omega$ can be divided into two $\star$-connected paths (ab) and (ba).[4] The Dobrushin-boundary conditions are defined to be $\oplus$ on (ab) and $\ominus$ on (ba).

The Ising model satisfies the domain Markov property. Suppose that $G' \subset G$ are two finite subgraphs of $\mathbb{Z}^d$. Fix $\beta > 0$ and some boundary conditions $b \in \{\ominus, \oplus\}^{\partial G}$. Let X be a random variable which is measurable with respect to vertices in G'. Then we have

$$\mu^b_{\beta,G}[X \,|\, \sigma_x = \psi_x, x \in G \setminus G'] = \mu^\psi_{\beta,G'}[X],$$

for all $\psi \in \{\ominus, \oplus\}^{G\setminus G'}$ such that $\psi = b$ on ∂G.

5.1 Infinite volume measures and phase transition

We have a natural partial order on the spin configurations $\{\ominus, \oplus\}^G$ given by

$$\sigma \le \sigma', \quad \text{if and only if} \quad \sigma_x \le \sigma'_x, \text{ for all } x \in G.$$

An event A is increasing if for any $\sigma \in A$ and $\sigma' \ge \sigma$, we have $\sigma' \in A$.

Theorem 5.1 (FKG Inequality). *Fix $\beta > 0$, a finite graph G and some boundary conditions b. For any two increasing events A and B, we have*

$$\mu^b_{\beta,G}[A \cap B] \ge \mu^b_{\beta,G}[A]\mu^b_{\beta,G}[B].$$

Proof. Similar as the proof of Theorem 3.1. □

Corollary 5.2 (Comparison between boundary conditions). *Fix $\beta > 0$, a finite graph G. For boundary conditions $b_1 \le b_2$ and any increasing event A, we have*

$$\mu^{b_1}_{\beta,G}[A] \le \mu^{b_2}_{\beta,G}[A].$$

We can then obtain the infinite-volume measures in the similar way as in Section 3.2.

[4]two vertices x and y are $\star$-neighbors if $\|x-y\|_\infty = 1$ and each vertex has eight neighbors in this metric.

Proposition 5.3. *Fix $\beta > 0$. There exist two (possibly equal) infinite-volume measures $\mu_\beta^\oplus$ and $\mu_\beta^\ominus$, called the infinite-volume Ising model with $\oplus$ and $\ominus$ boundary conditions respectively, such that for any event A depending on a finite number of sites,*

$$\lim_{n\to\infty} \mu_{\beta,\Lambda_n}^\oplus[A] = \mu_\beta^\oplus[A], \quad \lim_{n\to\infty} \mu_{\beta,\Lambda_n}^\ominus[A] = \mu_\beta^\ominus[A].$$

Proposition 5.4. *The measures $\mu_\beta^\oplus$ and $\mu_\beta^\ominus$ are translation invariant and are ergodic. For $\mu_\beta^\oplus$ or $\mu_\beta^\ominus$, there is no infinite cluster almost surely, or there exists a unique infinite cluster almost surely.*

Proof. The translation invariance and ergodicity can be proved in a similar way as in Section 3.2. We only need to explain the uniqueness of the infinite cluster. Exercise 5.2. □

The Ising model in infinite volume exhibits a phase transition.

Theorem 5.5. *Define*

$$\beta_c = \frac{1}{2}\log(1+\sqrt{2}).$$

We have $\mu_\beta^\oplus[\sigma_0] = 0$ when $\beta < \beta_c$, and $\mu_\beta^\oplus[\sigma_0] > 0$ when $\beta > \beta_c$.

We will derive Theorem 5.5 using the phase transition of the random-cluster model. To this end, we introduce the Edwards-Sokal coupling. Let G be a finite graph and ω be a configuration in $\{0,1\}^{E(G)}$. We construct a spin configuration $\sigma \in \{\ominus,\oplus\}^G$ by assigning independently to each cluster of ω a $\oplus$ or $\ominus$ spin with probability $1/2$ (note that all the vertices of a cluster receive the same spin).

Proposition 5.6. *Fix $p \in (0,1)$ and a finite graph G. If the configuration ω is sampled according to the random-cluster measure with parameters $(p,2)$ and free boundary conditions, then the spin configuration σ is distributed according to an Ising measure with free boundary conditions and inverse-temperature*

$$\beta = \frac{1}{2}\log\frac{1}{1-p}. \tag{5.1}$$

Proof. Denote by P the probability measure on (ω,σ) where ω is a random-cluster configuration with free boundary conditions and σ is the corresponding spin configuration as explained above. By the construction, we have

$$P[(\omega,\sigma)] = \frac{1}{\mathcal{Z}^0_{p,2,G}} p^{o(\omega)}(1-p)^{c(\omega)}2^{k(\omega)} \times 2^{-k(\omega)} = \frac{1}{\mathcal{Z}^0_{p,2,G}} p^{o(\omega)}(1-p)^{c(\omega)}.$$

Now we construct another probability measure Q on (ω,σ) as follows. Let σ be a spin configuration sampled according to the Ising model with free boundary conditions and the inverse-temperature β. We then define ω from σ by closing all edges between neighboring vertices with different spins and by independently opening edges between neighboring vertices with the same spin with probability p. Denote by $r(\sigma)$ the number of edges between neighboring vertices with different spins. By the construction, we have

$$Q[(\omega,\sigma)] \propto e^{-2\beta r(\sigma)}p^{o(\omega)}(1-p)^{c(\omega)-r(\sigma)}.$$

By the choice of β, we see $1-p = e^{-2\beta}$. Thus

$$Q[(\omega,\sigma)] \propto p^{o(\omega)}(1-p)^{c(\omega)}.$$

This implies that $Q = P$ and hence gives the conclusion. □

In Proposition 5.6, if ω is sampled according to the random-cluster measure with parameters $(p,2)$ and wired boundary conditions, the Edwards-Sokal coupling provides us with a Ising configuration with $\oplus$-boundary conditions (or $\ominus$-boundary conditions).

Corollary 5.7. *For $p \in (0,1)$, a finite graph G and β defined through* (5.1)*, we have*

$$\mu^f_{\beta,G}[\sigma_x\sigma_y] = \phi^0_{p,2,G}[x \leftrightarrow y], \quad \mu^{\oplus}_{\beta,G}[\sigma_x] = \phi^1_{p,2,G}[x \leftrightarrow \partial G].$$

Proof of Theorem 5.5. Combining Corollary 5.7 and Theorem 3.10. Moreover, we also have the exponential decay for the subcritical phase. □

Theorem 5.8.

- *For $\beta \le \beta_c$, we have $\mu^{\oplus}_\beta[\sigma_0] = 0$. Consequently, $\mu^{\oplus}_\beta = \mu^{\ominus}_\beta$ and this is the unique infinite-volume measure.*
- *For $\beta > \beta_c$, we have $\mu^{\oplus}_\beta[\sigma_0] > 0$. The set of infinite-volume measures is given by*

$$\{\lambda\mu^{\oplus}_\beta + (1-\lambda)\mu^{\ominus}_\beta : \lambda \in [0,1]\}.$$

Proof. From Corollary 5.7 and conclusions from random-cluster models in Section 3, we have

$$\mu^{\oplus}_{\beta,\Lambda_n}[\sigma_0] = \phi^1_{p,2,\Lambda_n}[0 \leftrightarrow \partial\Lambda_n] \to \begin{cases} 0, & p \le p_c; \\ \phi^1_{p,2}[0 \leftrightarrow \infty] > 0, & p > p_c. \end{cases}$$

Therefore, we have $\mu^{\oplus}_\beta[\sigma_0] = 0$ for $\beta \le \beta_c$ and $\mu^{\oplus}_\beta[\sigma_0] > 0$ for $\beta > \beta_c$.

When $\beta \le \beta_c$, it remains to show that $\mu^{\oplus}_\beta = \mu^{\ominus}_\beta$. To this end, we need to show that $\mu^{\oplus}_\beta[A] = \mu^{\ominus}_\beta[A]$ for any increasing event A depending only on finitely many vertices. Suppose A is measurable with respect to the status of vertices inside Λ_N and suppose $N \le n$. We construct a probability measure on the triple $(\omega, \sigma^{\oplus}, \sigma^{\ominus})$ in the following way. We first sample ω according to $\phi^1_{p,2,\Lambda_n}$. Denote by $\mathcal{C}_n$ the cluster in ω connected to $\partial\Lambda_n$. For $v \in \mathcal{C}_n$, we set $\sigma^{\oplus}_v = \oplus$ and $\sigma^{\ominus}_v = \ominus$; for $v \in \Lambda_n \setminus \mathcal{C}_n$, we set $\sigma^{\oplus}_v = \sigma^{\ominus}_v$ which is either $\oplus$ or $\ominus$ with equal probability 1/2. Then we find that $\sigma^{\oplus} \sim \mu^{\oplus}_{\beta,\Lambda_n}$ and $\sigma^{\ominus} \sim \mu^{\ominus}_{\beta,\Lambda_n}$. We have

$$0 \le \mu^{\oplus}_{\beta,\Lambda_n}[A] - \mu^{\ominus}_{\beta,\Lambda_n}[A] = \mathbb{P}[\sigma^{\oplus} \in A, \sigma^{\ominus} \notin A] \le \phi^1_{p,2,\Lambda_n}[\partial\Lambda_N \leftrightarrow \partial\Lambda_n] \to 0, \quad \text{as } n \to \infty.$$

Therefore, $\mu^{\oplus}_\beta[A] = \mu^{\ominus}_\beta[A]$ as desired.

When $\beta > \beta_c$, we have $\mu^{\oplus}_\beta[\sigma_0] = \phi^1_{p,2}[0 \leftrightarrow \infty] > 0$ and $\mu^{\ominus}_\beta[\sigma_0] = -\phi^1_{p,2}[0 \leftrightarrow \infty] < 0$. For the rest of the conclusion, see [Aiz80, Hig81]. □

5.2 High and low temperature expansions

Graph terminologies

For a graph $G = (V, E)$, define $\mathcal{E}_G$ to be the collection of even subgraph of G, i.e. the set of subgraphs ω of G such that every vertex in V is the end-point of an even number of edges of ω. Generally, for $A \subset V$, let $\mathcal{E}_G(A)$ be the set of subgraphs ω of G such that

- every vertex of $V \setminus A$ is the end-point of an even number of edges of ω,
- every vertex of A is the end-point of an odd number of edges of ω.

Note that if $\#A$ is odd, then $\mathcal{E}_G(A)$ is empty. The set $\mathcal{E}_G(\emptyset)$ is the same as $\mathcal{E}_G$.

High temperature expansion

The high temperature expansion of the Ising model is based on the following identity:

$$e^{\beta\sigma_x\sigma_y} = \cosh(\beta) + \sigma_x\sigma_y \sinh(\beta) = \cosh(\beta)(1 + \tanh(\beta)\sigma_x\sigma_y). \tag{5.2}$$

Proposition 5.9. *Let G be a finite graph and $\beta > 0$. We have*

$$\mathcal{Z}^f_{\beta,G} = 2^{\#V(G)} \cosh(\beta)^{\#E(G)} \sum_{\omega\in\mathcal{E}_G} \tanh(\beta)^{o(\omega)}. \tag{5.3}$$

Proof. By (5.2), we have

$$\begin{aligned}\mathcal{Z}^f_{\beta,G} = \sum_{\sigma}\prod_{(x,y)\in E} e^{\beta\sigma_x\sigma_y} &= \cosh(\beta)^{\#E}\sum_{\sigma}\prod_{(x,y)\in E}(1+\tanh(\beta)\sigma_x\sigma_y)\\ &= \cosh(\beta)^{\#E}\sum_{\sigma}\sum_{\omega\subset E}\tanh(\beta)^{o(\omega)}\prod_{(x,y)\in\omega}\sigma_x\sigma_y.\end{aligned}$$

Note that, for fixed $\omega \subset E$,

$$\sum_{\sigma}\prod_{(x,y)\in\omega}\sigma_x\sigma_y = \begin{cases}2^{\#V}, & \text{if } \omega\in\mathcal{E}_G,\\ 0, & \text{otherwise.}\end{cases}$$

This gives (5.3). □

Low temperature expansion

The low temperature expansion of the Ising model is a graph representation on the dual lattice. For each spin configuration $\sigma \in \{\oplus,\ominus\}^{V(G)}$, define the contour configuration $\omega[\sigma] \in \{0,1\}^{E(G^*)}$ on the dual lattice by the following, for each edge $e = (x,y) \in E(G)$,

$$\omega[\sigma](e^*) = \begin{cases}1, & \text{if } \sigma_x \neq \sigma_y,\\ 0, & \text{otherwise.}\end{cases}$$

The construction is one-to-one for any boundary conditions b, and it is two-to-one for free boundary conditions due to the symmetry. Once we know the boundary conditions b, one may reconstruct the spin configuration σ from the contour configuration $\omega[\sigma]$. If σ is sampled according to the Ising measure, then the probability of the corresponding contour configuration ω is proportional to $e^{-2\beta o(\omega)}$.

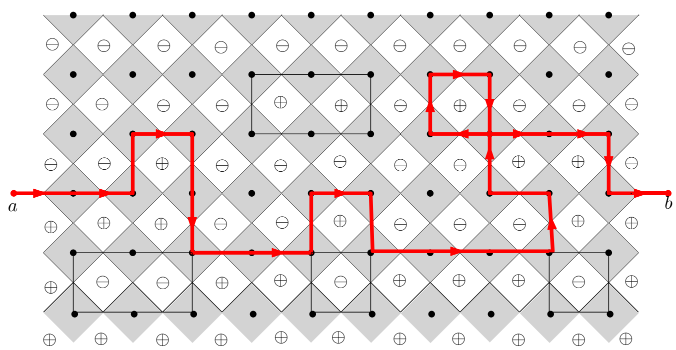


Figure 5.1: Low temperature expansion for Dobrushin boundary conditions.

Example 5.10. *Consider the Ising model on G with $\oplus$-boundary conditions, then the corresponding contour configuration ω is an even subgraph of G^*. We find*

$$\mathcal{Z}^{\oplus}_{\beta,G} = e^{\beta\#E(G^*)}\sum_{\omega\in\mathcal{E}_{G^*}} e^{-2\beta o(\omega)}.$$

Example 5.11. *Consider the Ising model on $(\Omega; a, b)$ (which is a subgraph of $\mathbb{Z}^2$) with Dobrushin boundary conditions, then the corresponding contour configuration is composed of loops togehter with one interface running between the two edges of $(\mathbb{Z}^2)^*$ at a and b, see Figure 5.1. In this case, we have*

$$\mathcal{Z}^{dobr}_{\beta,G} = e^{\beta\#E(G^*)}\sum_{\omega\in\mathcal{E}_{G^*\cup\{a,b\}}(\{a,b\})} e^{-2\beta o(\omega)}.$$

Krammers-Wannier duality

Proposition 5.12. *Let $\beta > 0$ and define $\beta^* \in (0, \infty)$ such that*

$$\tanh(\beta^*) = e^{-2\beta}. \tag{5.4}$$

Then for every graph G, we have

$$2^{\#V(G^*)} \cosh(\beta^*)^{\#E(G^*)} \mathcal{Z}^{\oplus}_{\beta,G} = e^{\beta \# E(G^*)} \mathcal{Z}^{f}_{\beta^*,G^*}.$$

Physicists expect the free energy to exhibit only one singularity, localized at β_c. If $\beta_c^* \neq \beta_c$, there would be two singularities. Thus $\beta_c = \beta_c^*$. Note that β_c in Theorem 5.5 is the unique solution to (5.4) for $\beta^* = \beta$.

5.3 Fermionic observable

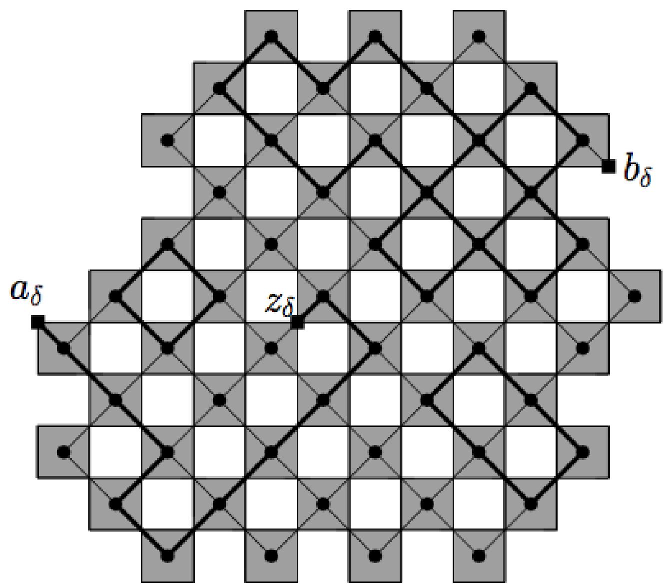


Figure 5.2

Let $z_\delta \in \Omega_\delta^\diamond$, let $\mathcal{E}(a_\delta, z_\delta)$ be the set of collections of contours drawn on Ω_δ composed of loops and one interface from a_δ to z_δ. For such a configuration $\omega \in \mathcal{E}(a_\delta, z_\delta)$, denote by $\gamma(\omega)$ the interface from a_δ to z_δ (turning left when there is ambiguity). See Figure 5.2. Note that, if we consider the critical Ising model on Ω_δ^* with Dobrushin boundary conditions, see Figure 5.3, then $\omega \in \mathcal{E}(a_\delta, b_\delta)$ is the low temperature expansion and γ_δ is the interface in ω connecting a_δ to b_δ.

Definition 5.13. *On a Dobrushin domain $(\Omega_\delta^\diamond; a_\delta, b_\delta)$, the spin-Ising fermionic observable at $z_\delta \in \Omega_\delta^\diamond$ is defined by*

$$F_\delta(z_\delta) = \frac{\sum_{\omega \in \mathcal{E}(a_\delta, z_\delta)} e^{-\frac{i}{2} W_{\gamma(\omega)}(a_\delta, z_\delta)} (\sqrt{2}-1)^{o(\omega)}}{\sum_{\omega \in \mathcal{E}(a_\delta, b_\delta)} e^{-\frac{i}{2} W_{\gamma(\omega)}(a_\delta, b_\delta)} (\sqrt{2}-1)^{o(\omega)}},$$

where the winding $W_\gamma(a_\delta, z_\delta)$ is the total rotation in radians of the curve γ between a_δ and z_δ.

Theorem 5.14. *Let $(\Omega; a, b)$ be a Dobrushin domain and assuming that $\partial\Omega$ is smooth in a neighbor of b. Then*

$$F_\delta(\cdot) \to \sqrt{\frac{\varphi'(\cdot)}{\varphi'(b)}}, \quad \textit{as } \delta \to 0, \quad \textit{locally uniformly},$$

where φ is any conformal map from Ω onto $\mathbb{H}$ sending a to ∞ and b to 0.

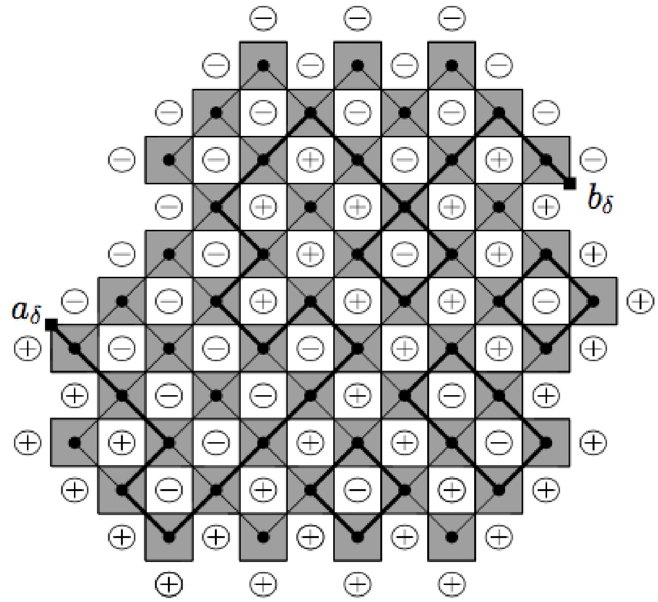


Figure 5.3

The proof has three steps:

- First, prove the s-holomorphicity of F_δ.
- Second, prove the convergence of the associated function H_δ.
- Third, prove the convergence of F_δ.

Proposition 5.15. *For $\delta > 0$, the fermionic observable F_δ is s-holomorphic on $\Omega_\delta^\diamond$.*

Proof. Let x, y be two adjacent medial vertices connected by the edge $e = (x, y)$. Without loss of generality, assume e is pointing southeast, thus $\ell(e) = \mathbb{R}$. Let v be the vertex of Ω_δ bordering the medial edge e. For $\omega \in \mathcal{E}(a_\delta, x_\delta)$, define

$$x_\omega = \frac{1}{\mathcal{Z}}(\sqrt{2}-1)^{o(\omega)} \exp\left(-\frac{i}{2}(W_{\gamma(\omega)}(a_\delta, x_\delta) - W_{\gamma_0}(a_\delta, b_\delta))\right),$$

where γ_0 is some fixed arbitrary path from a_δ to b_δ and $\mathcal{Z}$ is a normalizing real number not depending on the configuration. For $\omega \in \mathcal{E}(a_\delta, y_\delta)$, define y_ω similarly. We will show that

$$\sum_{\omega \in \mathcal{E}(a_\delta, x_\delta)} \Re(x_\omega) = \sum_{\omega' \in \mathcal{E}(a_\delta, y_\delta)} \Re(y_{\omega'}). \tag{5.5}$$

We will partition the set of configurations into pairs of configurations (ω, ω'), where ω contributes to $F_\delta(x_\delta)$ and ω' contributes to $F_\delta(y_\delta)$, such that $\Re(x_\omega) = \Re(y_{\omega'})$.

Consider the interface in ω that connecting a_δ to x_δ or y_δ, there are four cases. Case 1. The interface reaches x or y and can be extended by one step to reach the other one. Case 2. The interface reaches v before x or y, and can be extended by one step to x or y. Case 3. The interface reaches x or y and sees a loop preventing it from being extended to the other one. Case 4. The interface reaches x or y, then goes away from v and comes back to the other one. Note that, in cases 1(a), 1(b), 2, there can be a loop or even the past of the interface passing through v, but this does not change the computation.

We obtain the following table for x_ω and $y_{\omega'}$, see Figure 5.5. Moreover, we can also compute the argument modulo π of x_ω. Upon projecting on $\mathbb{R}$, the result follows. $\square$

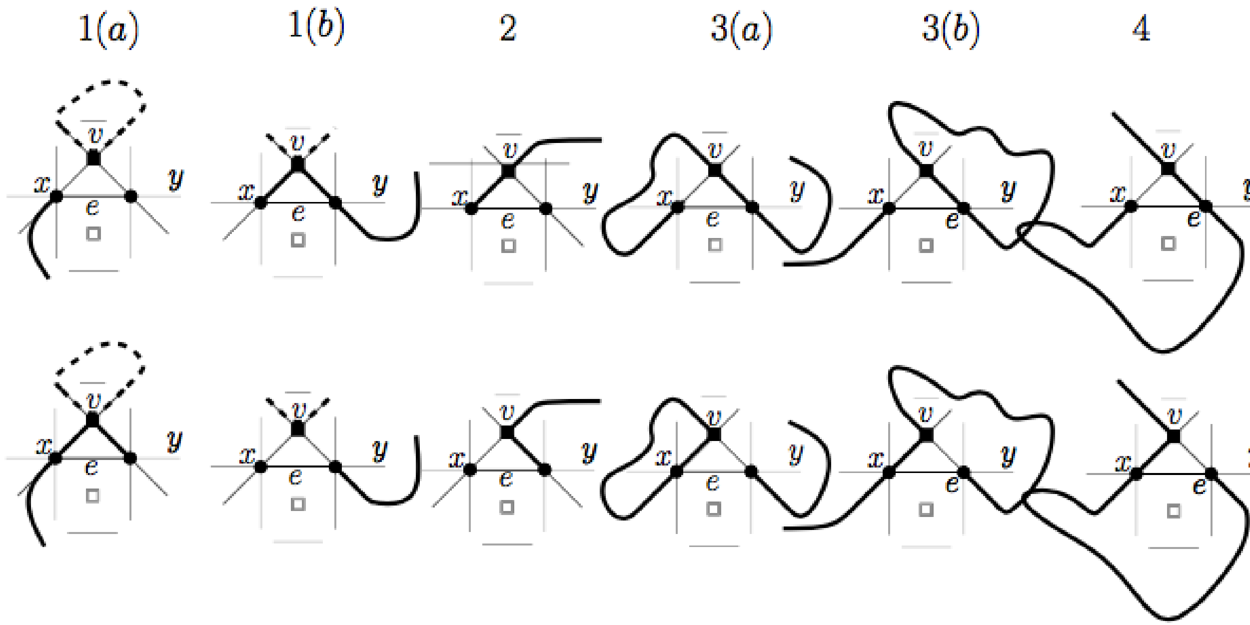


Figure 5.4

configuration	Case 1(a)	Case 1(b)	Case 2	Case 3(a)	Case 3(b)	Case 4
x_ω	x_ω	x_ω	x_ω	x_ω	x_ω	x_ω
$y_{\omega'}$	$(\sqrt{2}-1)e^{i\pi/4}x_\omega$	$\frac{e^{i\pi/4}}{\sqrt{2}-1}x_\omega$	$e^{-i\pi/4}x_\omega$	$e^{3i\pi/4}x_\omega$	$e^{3i\pi/4}x_\omega$	$e^{-5i\pi/4}x_\omega$
arg. x_ω mod π	$5\pi/8$	$\pi/8$	$\pi/8$	$5\pi/8$	$5\pi/8$	$5\pi/8$

Figure 5.5

Proof of Theorem 5.14 (sketch). Since F_δ is s-holomorphic, one can define the observable H_δ as in Theorem 4.8 with the requirement that it is equal to zero on the white face adjacent to b. Then one can show that away from a, the functions $(H_\delta)_{\delta>0}$ remains bounded (the proof is sophisticated where one uses the boundary modification trick). The boundedness implies precompactness of $(F_\delta)_{\delta>0}$ away from a.

Consider a convergent subsequence $(F_{\delta_n}, H_{\delta_n})$ converging to (F, H). One can check that H is equal to zero on $\partial\Omega \setminus \{a\}$ and it is harmonic inside Ω (not easy). Thus $H = \lambda\Im\varphi$ for some constant $\lambda > 0$. Thus $2\lambda\Im\varphi = \Im\int F^2$ which gives $F^2 = 2\lambda\varphi'$. Since $F(b) = 1$, we have $F = \sqrt{\varphi'/\varphi'(b)}$. □

5.4 Exercises

Exercise 5.1. *Denote by* $\mathcal{Z}^f_{Ising}(\beta, G)$ *the partition function for the Ising model with free boundary conditions and by* $\mathcal{Z}^0_{RCM}(p, 2, G)$ *the partition function for the random-cluster model with free boundary conditions. Assume* β *and* p *are related through* (5.1). *Show that*

$$\mathcal{Z}^f_{Ising}(\beta, G) = e^{\beta\#E(G)}\,\mathcal{Z}^0_{RCM}(p, 2, G).$$

Exercise 5.2. *Prove Proposition 5.4. Hint: Consider Ising model on* $\mathbb{Z}^2$, *we see that any* $\oplus$*-cluster is surrounded by a* $\star$*-connected* $\ominus$*-circuit.*

Exercise 5.3. *Use the high and low temperature expansions to show that* $\beta_c \in (0, \infty)$.

SRT 7. *Convergence of interfaces in Ising model.*

Please give a report on Ising model discussing the following topics.

(1) convergence of the interface to SLE(3) in Dobrushin domain.

(2) convergence of the interface with general boundary conditions.

References: [CS12, CDCH$^+$14, Izy15, Izy17].

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